

AD 615508

NAVWEPS REPORT 8611
NOTS TP 3635
COPY 73

THEORETICAL INVESTIGATION OF THE DYNAMIC BEHAVIOR OF THE MERCURY DAMPER

by

Prescott D. Crout

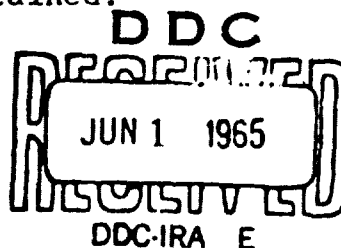
Research Department

| | | | | |
|------------|-------------|----|---|-----|
| COPY | 2 | OF | 3 | 126 |
| HARD COPY | \$. 4 . 00 | | | |
| MICROFICHE | \$. 1 - 00 | | | |

126P

ABSTRACT. The dynamic behavior of the mercury damper is investigated. Particular attention is paid to the eccentric annular mercury configuration, which is the final continuous ring phase which occurs in the operation of all mercury dampers. In this phase the damping is the poorest, and the system is closely linear. The crescent, or broken-ring, case is also considered.

During the course of the investigation the hydrodynamic problem is treated as three dimensional, and extensive use is made of a variational principle of least viscous frictional power loss, which is derived. A variational principle of least constraint is also used to advantage. Formulas for calculating the behavior of the mercury damper are obtained.



U. S. NAVAL ORDNANCE TEST STATION

China Lake, California

November 1964

ARCHIVE COPY

U. S. NAVAL ORDNANCE TEST STATION

AN ACTIVITY OF THE BUREAU OF NAVAL WEAPONS

J. I. HARDY, CAPT., USN
Commander

WM. B. McLEAN, Ph.D.
Technical Director

FOREWORD

The purpose of this work is to investigate the dynamic behavior of the mercury damper. Formulas are obtained for calculating the behavior of the damper. The next phase of the work should be a comparison of the results of such calculations with the corresponding results obtained experimentally.

Work on this report was performed during the period September 1961 and October 1963 under Bureau of Naval Weapons Task Assignments RTAD 20-001/216-1/F019-03-002 and RT 70-43-001/216-1/F019-03-002.

The report was reviewed for adequacy and technical accuracy by Roy Dale Cole of the Aviation Ordnance Department.

Released by
GILBERT J. PLAIN,
Acting Head,
Research Department
12 August 1964

Under authority of
WM. B. McLEAN
Technical Director

NOTS Technical Publication 3635
NAVWEPS Report 8611

Published by.....Research Department
Collation.....Cover, 63 leaves, abstract cards
First printing.....155 numbered copies
Security classification.....UNCLASSIFIED

CONTENTS

| | |
|---|-----|
| Introduction | 1 |
| Mechanical System. Statement of Problem | 2 |
| Action of the Damper Upon the Main Spinning Body | 7 |
| Relation of the Viscous Frictional Power Loss to the Torque | .15 |
| Minimum Principle for the Viscous Frictional Power Loss | .17 |
| Improved Treatment of the Action of the Damper on the Main Spinning Body | .25 |
| Eccentric Annular Configuration--Nature of the Free Surface of the Mercury. Specification of the Shape of the Velocity Field. | .28 |
| Eccentric Annular Configuration--Magnitude of the Velocity Field | .34 |
| Calculation of the Viscous Frictional Power Loss P for Given b and $q(\lambda, z)$ | .38 |
| Determination of $q(\lambda, z)$ | .49 |
| Determination of the Viscous Frictional Power Loss P | .53 |
| Extension of the Minimum Principle for the Viscous Frictional Power Loss. | .54 |
| Principle of Least Constraint. | .56 |
| Determination of the Pressure Field. | .57 |
| Surface Analogue of the Principle of Least Constraint. Determination of b and c_0 | .64 |
| Application of the Principle of Least Surface Con- straint to the Determination of b and c_0 | .65 |

| | |
|--|-----|
| Determination of the Damping Constant and Precessional Angular Velocity | 95 |
| Application of Procedures for Linear Differential Equations | 98 |
| Polar Form of the Torque Differential Equations | 102 |
| Case Where the Mercury Configuration is Crescent-Shaped. | 104 |
| Appendixes: | |
| A. Effect of the Tilt and Three Dimensional Nature of the Damper on the Force Which it Exerts on the Main Spinning Body. | 111 |
| B. Outline of the Procedure for Solving a Problem. | 121 |

INTRODUCTION

The purpose of the present investigation is to examine the dynamic behavior of the mercury damper, and to devise means for calculating this behavior. Since mercury dampers can operate in several different ways, depending upon the mercury configuration, the nature of the inner free surface, and the state of turbulence; and since any given damper operates in at least two very different manners, the overall behavior of a mercury damper may appear to be both complex and peculiar.

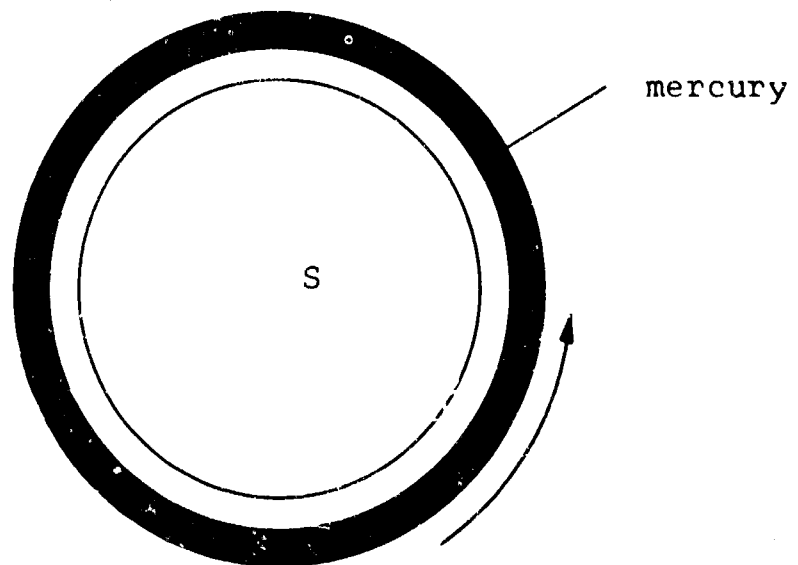
It will be seen that any mercury damper acts more effectively when the mercury forms a broken ring than when it forms a continuous ring. It follows that the transition from a broken ring to a continuous ring, which occurs in any case during the course of the damping, corresponds to a transition from a more effective to a less effective action of the damper. It is particularly important that the behavior of the damper in this unfavorable "continuous ring" case be analyzed; and a considerable portion of this report is devoted to this analysis. The more favorable broken ring case is also treated; however, it is very much simpler than the unfavorable case just mentioned.

It will be shown that the viscous frictional power loss in the mercury is of primary importance in eliminating the precession, or wobble; also that this loss satisfies a minimum principle which constitutes a powerful tool for determining the velocity distribution in the mercury, and hence the power loss itself. The pressure distribution is determined using another variational principle--the principle of least constraint--which is devised for this purpose.

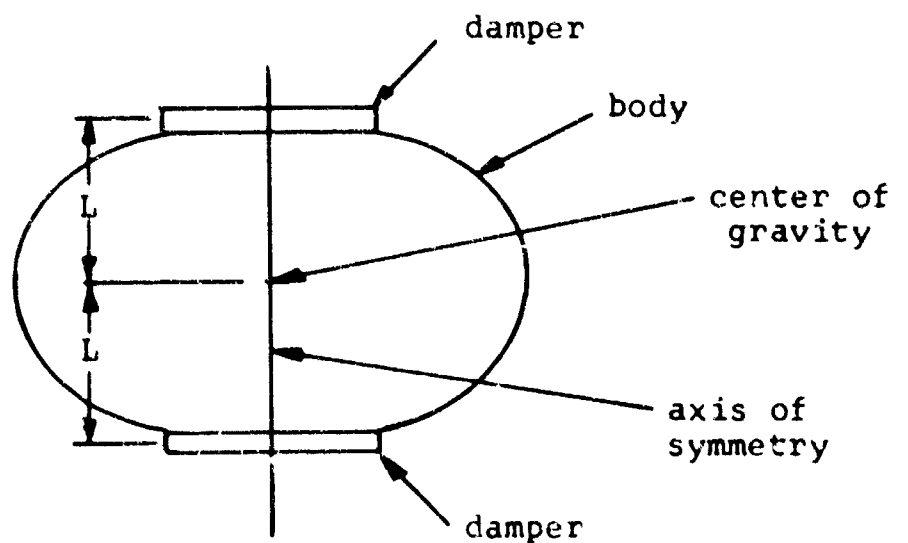
It will be shown that when the wobble is small and the mercury is in the unfavorable continuous ring configuration the system is approximately linear, and its behavior is described by linear differential equations with constant coefficients. The usual methods for handling such equations can hence be applied. This is not true when the wobble is larger and the mercury is in the broken ring configuration. In both cases, however, specific formulas are obtained for calculating the behavior of the system (see Appendix B).

MECHANICAL SYSTEM. STATEMENT OF PROBLEM

The mercury damper consists of a circular channel, or race, partially filled with mercury, as shown in the figure below. Two such dampers are mounted with their planes



perpendicular to the axis of symmetry of the main spinning body, and with their centers lying on this axis on opposite sides of the center of gravity of this body and equally spaced from it, as shown in the following figure. By using two



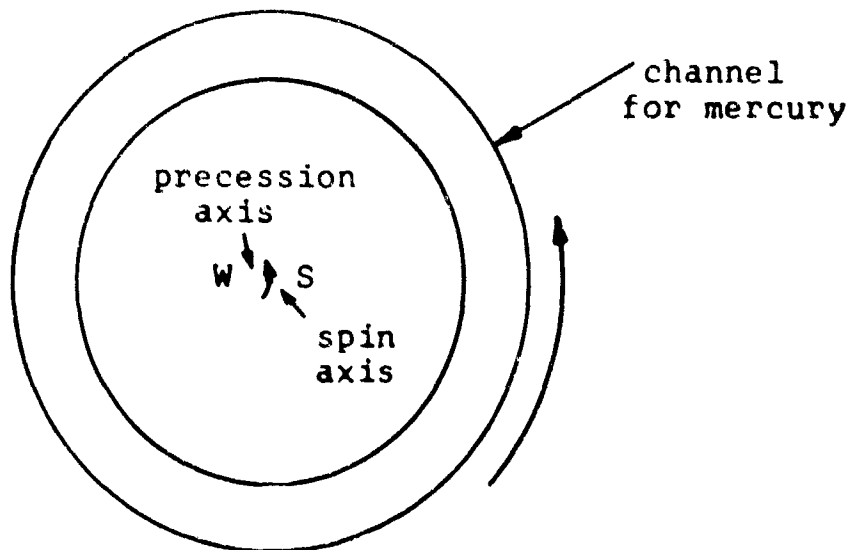
dampers the translational effects on the main spinning body cancel, whereas the rotational effects add. We shall suppose that the ratio of the principal moments of inertia of the main body at its center of gravity and the initial motion given to this body are such that this body precesses about an axis which passes through the center of gravity and approximates the spin axis, the angular velocity of precession exceeding that of spin. More exactly let

$$\begin{aligned}
 A &= \text{moment of inertia of the main body about its axis of symmetry,} \\
 B &= \text{moment of inertia of the main body about a transverse axis through its center of gravity,} \\
 \nu &= \text{angular velocity of spin,} \\
 \beta &= \text{angular velocity of precession;}
 \end{aligned}
 \tag{1}$$

then

$$\frac{A}{B} > 1 \quad \text{and} \quad \beta = \frac{A\nu}{B} \text{ closely.} \tag{2}$$

It will evidently suffice to consider one of the two dampers; hence looking down upon the upper damper shown in the figure at the bottom of page 2 the channel, or race, containing the mercury appears as shown below. The center of the channel is S, which point is also the point of intersection of the spin axis with the plane of the damper. The channel rotates



about point S with angular velocity ν ; however, in addition point S rotates about W, the point of intersection of the precession axis with the plane of the damper, with angular velocity β . We have seen that $\beta > \nu$.

Let us observe the damper from a reference system which rotates about the precession axis with angular velocity β counterclockwise; then the velocity vector V and the acceleration vector A of any particle of mercury will be composed of components as follows:

$$V = V_R + V_L, \quad (3)$$

$$A = A_R + A_L + A_C \quad (4)$$

where

$$\begin{aligned} V_R \text{ and } A_R & \text{ are the relative velocity and} \\ & \text{relative acceleration vectors,} \\ & \text{respectively, seen while riding} \\ & \text{with the moving reference system,} \\ V_L \text{ and } A_L & \text{ are the locked velocity and} \\ & \text{locked acceleration vectors, re-} \\ & \text{spectively, which the particle} \\ & \text{would have if it were locked with} \\ & \text{the moving reference system,} \\ A_C & = 2\vec{\beta} \times V_R = \text{Coriolis acceleration vector,} \\ \vec{\beta} & = \text{precessional angular velocity vector.} \end{aligned} \quad (5)$$

The acceleration components A_L and A_C can evidently be replaced in effect by two systems of inertia forces of intensities $(-A_L)$ and $(-A_C)$ per unit mass, respectively. Our problem thus becomes one in which we may regard ourselves as being stationary, and the velocity and acceleration of a mercury particle as being given by V_R and A_R , respectively. The main body now apparently rotates clockwise about the spin axis, which passes through the stationary point S, with an angular velocity $(\beta - \nu)$. Two body forces are acting: the centrifugal force of intensity $(-A_L)$ per unit mass, and a "Coriolis force" of intensity $(-A_C)$.

The centrifugal force, which corresponds to the locked acceleration, is

$$\text{Centrifugal force} = r\beta^2 l_r \text{ per unit mass} \quad (6)$$

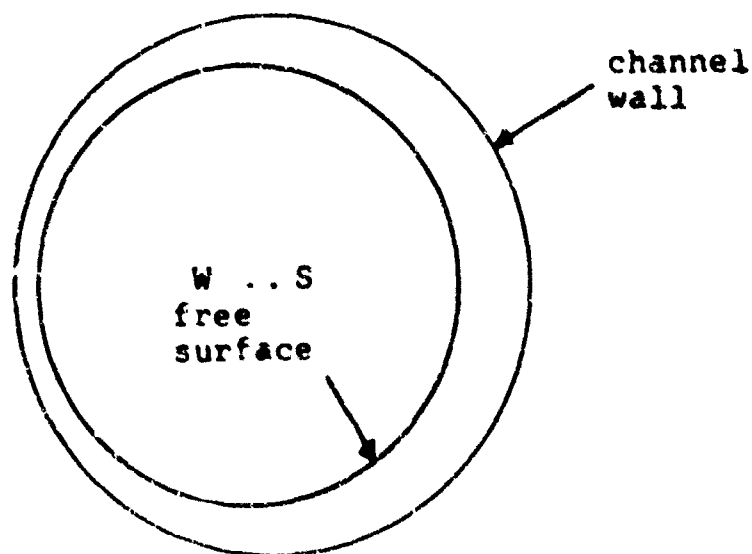
where r is the radial distance from W , and \mathbf{l}_r is a unit vector pointing radially outward from W . Equation 6 is evidently the negative gradient of the potential function

$$\Phi = -\frac{1}{2} r^2 \beta^2 \text{ per unit mass.} \quad (7)$$

The Coriolis force, which corresponds to the Coriolis acceleration, is

$$\text{Coriolis force} = -2\vec{\beta} \times \mathbf{V}_R \text{ per unit mass.} \quad (8)$$

Let us temporarily neglect the friction between the mercury and the channel; then the mercury will appear to be at rest, and $\mathbf{V}_R = 0$, $\mathbf{A}_R = 0$. The Coriolis force field therefore vanishes, and the mercury is at rest under the influence of the centrifugal force field alone. Since the force acting on a particle of mercury which lies on the free surface must act normal to this surface it follows from Eq. 6, or from Eq. 7 and the fact that the surface must be one of constant Φ , that r is constant on this surface. Seen from above, the free surface thus appears to be circular. For sufficiently small values of the distance WS the mercury is shaped like an eccentric cam, the outer circular boundary being centered on S , and the inner circular boundary being centered on W as shown in the figure below. The outer boundary is the channel wall; the inner boundary is a free surface, the radius of which is dictated by the amount of mercury in the damper.



As \overline{WS} is increased a situation arises wherein the two circles become tangent, after which the mercury becomes crescent-shaped. Since the total volume of mercury remains constant, the radius of the inner circle increases after the mercury has become crescent-shaped.

Other mercury configurations will be obtained if the inner radius of the channel is so large that it forms part of the inner boundary of the mercury surface, which could happen in the eccentric annular shape, the crescent shape, or both. We thus have four possible cases, which arise from the fact that for either the eccentric annular or the crescent configuration the inner surface of the mercury can be either entirely free or only partially free.

Next let us suppose that the friction between the mercury and the channel, which was neglected above, is now "gradually turned on;" then since the channel is apparently rotating clockwise with an angular velocity $(\beta - \nu)$ the mercury tends to be dragged clockwise. If the mercury is in a crescent configuration the channel rotates clockwise relative to the mercury body as a whole with an angular velocity $(\beta - \nu)$, and the center of gravity of the mercury will shift somewhat off of the extension of line WS in the clockwise sense.

If the mercury is in an eccentric annular configuration the mercury flows in the clockwise sense, the result being to reduce the velocity of the mercury relative to the channel. The mercury is dragged clockwise, and its axis of symmetry shifts from the line WS in the clockwise sense. Supposing that the inner free surface of the mercury can still be represented with sufficient accuracy by a circle, it follows that I , the center of this circle, shifts from W to a new position which lies somewhat above the line WS .

The various mercury configurations described above in the case of no friction are still possible with friction acting. In addition the mercury flow can in each case be either laminar or turbulent. In view of this and the fact that the mercury may be either eccentric annular or crescent shaped, and the inner mercury surface may be either entirely free or only partially free, we see that the mathematical analysis of the mercury damper involves the consideration of eight distinct cases. Which of these cases will exist at any time depends upon ν , A/B , \overline{WS} , the inner and outer radii of the channel, and the volume of mercury in the damper. In any given design A/B , the channel dimensions, and the volume of mercury are fixed; also ν is approximately fixed. As the damper functions, however, \overline{WS} decreases from its initial value to zero, and the velocity of the mercury relative to the channel drops to zero.

During this time all eight of the above cases will not be experienced, but three may be. In all damper designs the last case will be that in which the mercury shape is eccentric annular, the inner mercury surface entirely free, and the flow laminar. Preceding this may be one wherein the mercury shape is eccentric annular, the inner mercury surface either entirely or partially free, and the flow turbulent. Finally, preceding this may be one wherein the mercury is crescent-shaped, the inner surface but partially free, and the flow turbulent.

ACTION OF THE DAMPER UPON THE MAIN SPINNING BODY

We shall now calculate the force and torque exerted by the mercury upon the channel walls, and hence upon the main spinning body. In so doing we shall view the system while rotating counterclockwise with angular velocity β , as described in the previous section. We thus see the relative velocity V_R and the relative acceleration A_R ; and have a centrifugal force field given by Eqs. 6 or 7, and a Coriolis force field given by Eq. 8. Let

- F = total force exerted by the channel walls upon the mercury,
- F_L = total centrifugal force exerted upon the mercury,
- F_C = total Coriolis force exerted upon the mercury, (9)
- V_{RO} = relative velocity of the center of gravity of the mercury,
- A_{RO} = relative acceleration of the center of gravity of the mercury,
- M = total mass of mercury;

then for translational equilibrium of the mercury mass taken as a whole we have

$$F + F_L + F_C = M A_{RO}. \quad (10)$$

Although we may see the mercury as flowing in some way, its center of gravity will appear to be fixed; hence $A_{RO} = 0$ and Eq. 10 becomes

$$F = - F_L - F_C. \quad (11)$$

Multiplying Eq. 6 by dm , the element of mass, and integrating over the entire mass of mercury, we obtain¹

$$F_L = \beta^2 \int_M r l_r dm. \quad (12)$$

But \vec{r}_0 , the vector from W to the center of gravity of the mercury is given by

$$\vec{r}_0 = \frac{1}{M} \int_M r l_r dm. \quad (13)$$

Equation 13 in Eq. 12 gives

$$F_L = M \beta^2 \vec{r}_0, \quad (14)$$

which is identical with the centrifugal force that would be obtained if the entire mass were concentrated at the center of gravity.

Turning to the Coriolis force F_C we obtain from Eq. 8

$$F_C = - 2 \int_M \vec{\beta} \times V_R dm = - 2\vec{\beta} \times \int_M V_R dm. \quad (15)$$

But, noting Eq. 9,

$$\int_M V_R dm = \text{total momentum of mercury} = M V_{RO} = 0, \quad (16)$$

since $V_{RO} = 0$ due to the fact that the center of gravity of the mercury appears to be stationary. Equation 16 in Eq. 15 gives

$$F_C = 0. \quad (17)$$

Equations 14 and 17 in Eq. 11 now give finally

$$F = - M \beta^2 \vec{r}_0. \quad (18)$$

¹ See Appendix A.

Next, we shall determine the torque about point W that is exerted by the mercury on its channel walls. Let

τ = torque vector about W of the forces which the channel exerts on the mercury,

τ_L = torque vector about W of the centrifugal forces exerted on the mercury, (19)

τ_C = torque vector about W of the Coriolis forces exerted on the mercury;

then for rotational equilibrium of the mercury mass taken as a whole we have

$$\tau + \tau_L + \tau_C = \frac{d}{dt} \int_M \vec{r} \times \vec{V}_R dm \quad (20)$$

since W is a fixed point. Here the integral is the moment of momentum, or angular momentum, and \vec{r} is a vector from W to the elementary mass dm. Despite the fact that the mercury appears to be flowing, the moment of momentum remains constant; hence the right-hand side of Eq. 20 vanishes, and Eq. 20 becomes

$$\tau = -\tau_L - \tau_C. \quad (21)$$

Noting Eq. 6 we see that the centrifugal force acting on any elementary mass dm has a line of action which passes through the precession axis, and hence has a zero moment about this axis. Integrating over the entire mass it follows that

$$\tau_L = 0. \quad (22)$$

Finally it follows from Eq. 8 that

$$\tau_C = -2 \int_M \vec{r} \times (\vec{\beta} \times \vec{V}_R) dm, \quad (23)$$

whence

$$\tau_C = -2 \int_M [(\vec{r} \cdot \vec{V}_R) \vec{\beta} - (\vec{r} \cdot \vec{\beta}) \vec{V}_R] dm. \quad (24)$$

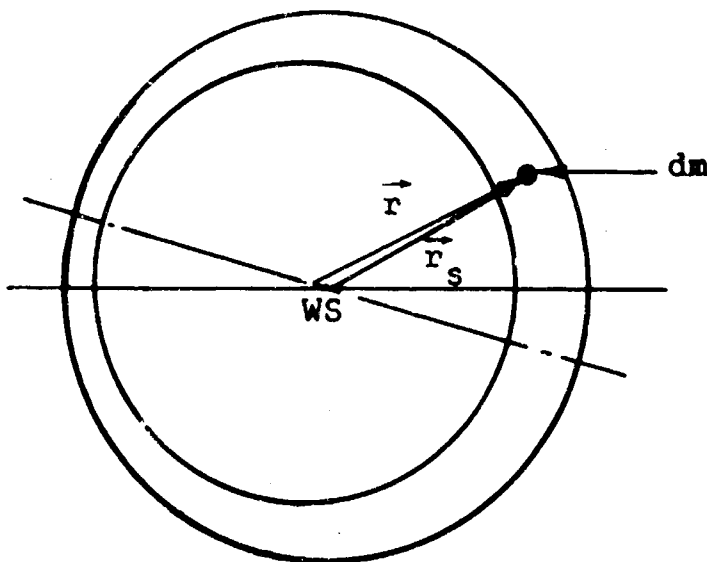
Since \vec{r} and $\vec{\beta}$ are orthogonal $\vec{r} \cdot \vec{\beta} = 0$, and the second term in the integrand of Eq. 24 vanishes, leaving

$$\tau_C = - 2\beta \int_M \vec{r} \cdot \vec{V}_R dm. \quad (25)$$

Noting the figure below let us place

$$\vec{r} = \vec{WS} + \vec{r}_s \quad (26)$$

where \vec{WS} is the vector from W to S, and \vec{r}_s is the vector



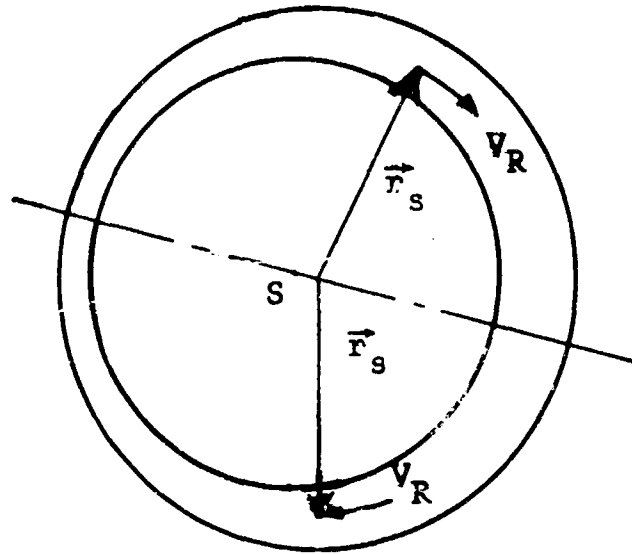
from S to the elementary mass dm . Substituting Eq. 26 in Eq. 25 we obtain

$$\begin{aligned} \tau_C &= - 2\beta \int_M (\vec{WS} + \vec{r}_s) \cdot \vec{V}_R dm \\ &= - 2\beta \vec{WS} \cdot \int_M \vec{V}_R dm - 2\beta \int_M \vec{r}_s \cdot \vec{V}_R dm; \end{aligned}$$

however the first term vanishes due to Eq. 16, leaving

$$\tau_C = - 2\beta \int_M \vec{r}_s \cdot \vec{V}_R dm. \quad (27)$$

For either the eccentric annular shape or the crescent shape, and for either laminar flow or turbulent flow the radial component of \vec{V}_R at any element dm is the negative of that component at the element dm which is located symmetrically opposite that plane of symmetry of the mercury which contains the spin axis, shown in the following figure. It follows



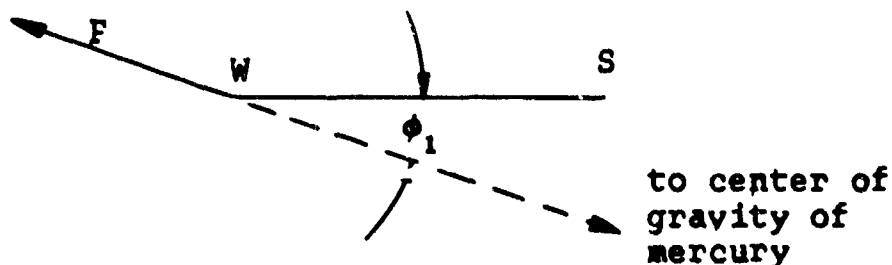
that the contribution of the mercury which lies on one side of this plane to the integral in Eq. 27 cancels the contribution of the mercury which lies on the other side; hence the integral vanishes, and

$$\tau_C = 0. \quad (28)$$

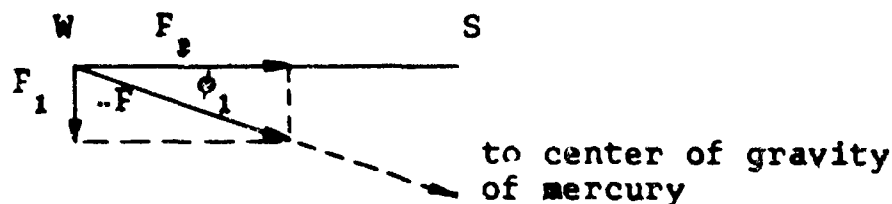
Equations 22 and 28 in Eq. 21 now give

$$\tau = 0. \quad (29)$$

We thus see that the moment about the precession axis of the forces exerted by the channel upon the mercury is zero. It follows that the resultant of these forces passes through W, and hence may be considered to act at W, as shown below.



F is the force exerted by the channel on the mercury; hence $(-F)$ is the force exerted by the mercury on the channel, and thereby on the main spinning body, as shown in the following figure. $(-F)$ may be replaced by the two component vectors



F_1 and F_2 , the magnitudes of which are, noting Eq. 18,

$$|F_1| = |F| \sin \phi_1 = M r_0 \beta^2 \sin \phi_1 = -M y_0 \beta^2, \quad (30)$$

$$|F_2| = |F| \cos \phi_1 = M r_0 \beta^2 \cos \phi_1 = M x_0 \beta^2$$

where x_0 and y_0 are the x and y coordinates of the center of gravity of the mercury measured with reference to a Cartesian coordinate system having W as an origin and WS as the positive x direction. Let

$$\overline{WS} = \xi; \quad (31)$$

then the moment of F_1 about the spin axis is, noting Eq. 30,

$$\begin{aligned} &\text{Moment exerted} \\ &\text{by the mercury} \\ &\text{about the spin} \\ &\text{axis} \end{aligned} = \tau_z = |F_1| \xi = -M y_0 \xi \beta^2 \quad (32)$$

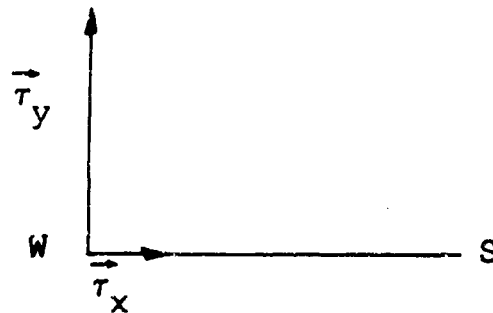
counterclockwise.

This acts in the same sense as ν , and, together with an equal moment due to the other damper, results in a rate of increase of ν given by

$$\dot{\nu} = 2 \frac{\tau_z}{A} = - \frac{2M y_0 \xi \beta^2}{A}. \quad (33)$$

It should be noted that y_0 is negative.

F_1 and F_2 give rise to torque vector components $\vec{\tau}_x$ and $\vec{\tau}_y$, respectively, about the center of gravity of the main spin-ning body as shown in the following figure, where noting the figure at the bottom of page 2,



$$\begin{aligned} |\vec{\tau}_x| &= |F_1|L, \\ |\vec{\tau}_y| &= |F_2|L, \end{aligned} \quad (34)$$

L being the distance, closely, between W and the center of gravity of the main spinning body. Since the torque vector of the external forces acting on this body about its center of gravity is the velocity of the tip of its angular momentum vector H for the center of gravity, we see that the torque vector components $\vec{\tau}_x$ and $\vec{\tau}_y$, when multiplied by 2 to include the effect of the other damper, may be regarded as velocity components of the tip of this vector. Since the H vector lies along the precession axis and is of magnitude $A\nu$, closely, we see, noting Eqs. 30 and 34, that $\vec{\tau}_x$ causes W to move toward S with a speed

$$\begin{aligned} \text{Velocity component of W toward S} &= -\dot{\xi} = \frac{2|\vec{\tau}_x|L}{A\nu} \\ &= -\frac{2My_0\beta^2L^2}{A\nu}. \end{aligned} \quad (35)$$

It is this velocity that lines up the precession and spin axes, and hence gets rid of the wobble.

Similarly, we see that $\vec{\tau}_y$ causes W to move in the y direction with a speed

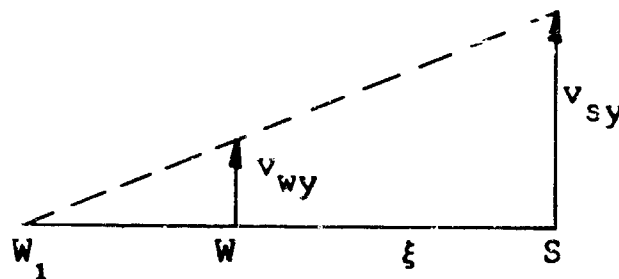
$$\text{Velocity component of W in y direction} = \frac{2|\vec{\tau}_y|L}{A\nu} = \frac{2Mx_0\beta^2L^2}{A\nu}, \quad (36)$$

the factor 2 being put in to include the effect of the other damper. This subtracts directly from the y component of the velocity of S relative to W, the y component of the velocity of S being due to the precession and equal to $\xi\beta$. The effect of this is to reduce the precessional angular velocity by the amount

$$\text{Reduction in precessional angular velocity} = \frac{2Mx_0\beta^2 L^2}{Av\xi} \quad (37)$$

since the angular velocity of S about W is the precessional angular velocity.

Another effect of the velocity component of W in the y direction, given by Eq. 36, is to shift the precession axis off of the H vector for the main body. Noting the figure below we see that although the H vector still passes through W, the precession axis now penetrates the plane of the damper



at W_1 , where at any instant

$$\overline{WW_1} = \frac{v_{wy}\xi}{v_{sy} - v_{wy}}. \quad (38)$$

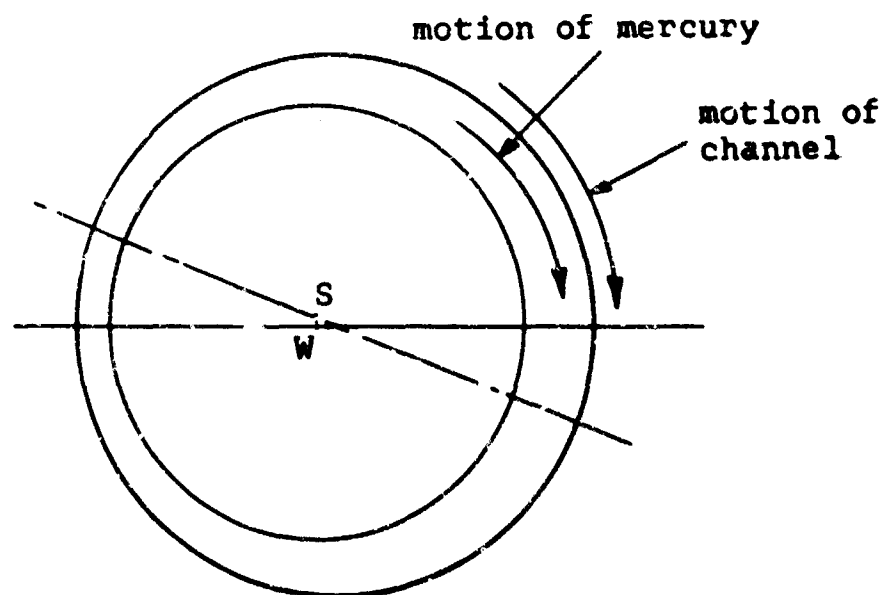
Here v_{wy} is given by Eq. 36, and

$$\overline{WS} = \xi, \quad v_{sy} = \xi\beta.$$

Attention has already been called to the fact that the translational effects of the two dampers cancel.

RELATION OF THE VISCOUS FRICTIONAL POWER LOSS TO THE TORQUE

The force F exerted by the channel on the mercury (see figure at the bottom of page 11) arises through the action of the stresses transmitted by the channel walls to the mercury. These stresses consist of the normal stress, or pressure, and the shear stress, which is due to the viscosity of the liquid. Taking moments about the spin axis, which passes through S , we see that since the pressure contributes nothing, the entire moment is contributed by the shear stress. Noting Eq. 32 and the figure on page 12 we see that this moment is of magnitude $|F_1|\xi$, and, looking down from the top, acts clockwise on the mercury (figure below). Here the eccentric annular shape is shown merely to fix ideas. At present



we are making no restriction upon the shape of the mercury.

Since the main body rotates with an apparent angular velocity $(\beta - \nu)$ clockwise, it delivers a power

$$P = |F_1|\xi (\beta - \nu) = -M y_0 \xi \beta^2 (\beta - \nu) \quad (39)$$

to the mercury. Although the mercury is apparently flowing, its kinetic energy remains constant; the potential energy

which it has by virtue of the potential function, Eq. 7, which arose because of the centrifugal force, is constant; and the power delivered by the Coriolis force field is zero, since the Coriolis force is everywhere perpendicular to the velocity of the mercury.² It follows that the entire power (Eq. 39) is dissipated as heat, and is equal to the viscous frictional power loss in the mercury. Substituting Eq. 39 in Eq. 35 we obtain

$$\begin{array}{l} \text{Velocity component} \\ \text{of } W \text{ toward } S \end{array} = - \dot{\xi} = \frac{2PL^2}{Av\xi(\beta-\nu)}; \quad (40)$$

hence the viscous friction loss P is the primary factor in the determination of $\dot{\xi}$, and hence in the elimination of the wobble.

² Instead of using the constancy of the potential energy to show that the power delivered by the centrifugal force field to the mercury is zero, we could also compute this power directly; thus noting Eqs. 6 and 7 we have

$$\begin{array}{l} \text{Power delivered to} \\ \text{mercury by centrifugal} \\ \text{force field} \end{array} = - \int_V \rho \mathbf{V}_R \cdot \nabla \phi dv = \rho \int_V [\phi \nabla \cdot \mathbf{V}_R - \nabla \cdot (\phi \mathbf{V}_R)] dv$$

where ρ is the density, and the integral extends throughout the volume of the mercury. Applying the divergence theorem, and noting that $\nabla \cdot \mathbf{V}_R = 0$ since the fluid is incompressible, this becomes

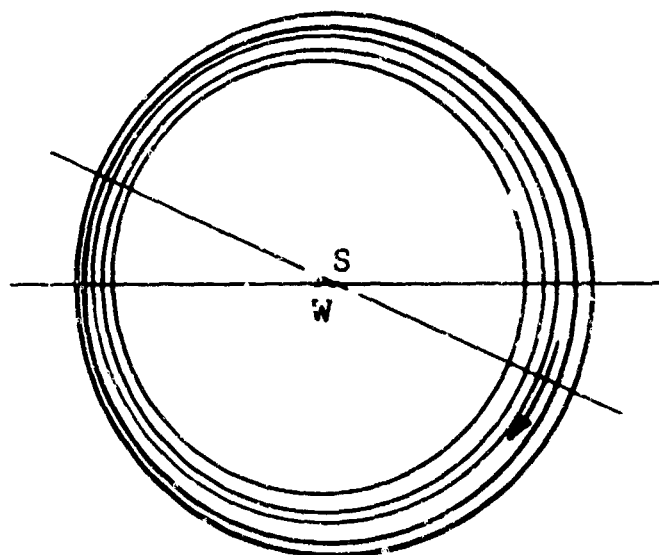
$$\begin{array}{l} \text{Power delivered to} \\ \text{mercury by centrifugal} \\ \text{force field} \end{array} = - \rho \int_a \phi \mathbf{V}_R \cdot \overline{da} = 0$$

where \overline{da} is the vector element of area, and the integral extends over the mercury surface; for $\mathbf{V}_R \cdot \overline{da} = 0$ because at this surface \mathbf{V}_R is tangent to the surface, and hence perpendicular to the normal.

The fact that in the case of the eccentric annular shape the mercury surface is not simply connected causes no difficulty in applying the divergence theorem, for a plane cross-cut could be introduced to render the surface simply connected without altering the values of the integrals.

MINIMUM PRINCIPLE FOR THE VISCOUS FRICTIONAL POWER LOSS

Since the mercury is incompressible $\nabla \cdot \mathbf{V}_R = 0$, and at any instant the lines of flow cannot terminate anywhere, but must be closed. It follows that at any instant the flow may be considered to be composed of an infinite number of closed elementary tubes of flow, figure below. Let us suppose that



the flow is stationary--that is, the flow pattern does not change with time, it being remembered that while observing this pattern we are rotating counterclockwise with an angular velocity β . We shall now show that if the flow is stationary, and if the shape of the velocity field is held fixed, so that the lines and tubes of flow are unaltered, then the flow will distribute itself among the various tubes in such a way that the viscous frictional power loss is a minimum.

Let us choose a set of rectangular xyz-coordinate axes to measure the apparent motion of the mercury; and let v_x , v_y , and v_z denote the x, y, and z components of \mathbf{V}_R , respectively; then the normal and shear stresses in the liquid which arise due to viscosity are, respectively,³

³ Rouse, Hunter. Fluid Mechanics for Hydraulic Engineers. New York, McGraw-Hill, 1938. Pp. 144-45.

$$\begin{aligned}\sigma_x &= 2\mu \frac{\partial v_x}{\partial x}, \quad \sigma_y = 2\mu \frac{\partial v_y}{\partial y}, \quad \sigma_z = 2\mu \frac{\partial v_z}{\partial z}, \\ \tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right), \quad \tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right), \\ \tau_{zx} &= \tau_{xz} = \mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right).\end{aligned}\quad (41)$$

The corresponding time rates of change of strain are, respectively,

$$\begin{aligned}\dot{\epsilon}_x &= \frac{\partial v_x}{\partial x}, \quad \dot{\epsilon}_y = \frac{\partial v_y}{\partial y}, \quad \dot{\epsilon}_z = \frac{\partial v_z}{\partial z}, \\ \dot{\gamma}_{xy} &= \dot{\gamma}_{yx} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}, \quad \dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}, \\ \dot{\gamma}_{zx} &= \dot{\gamma}_{xz} = \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}.\end{aligned}\quad (42)$$

Noting that the fluid is incompressible the viscous frictional power loss per unit volume is hence

$$\begin{aligned}\text{Power loss per unit volume} &= 2\mu \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] \\ &+ \mu \left[\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)^2 \right].\end{aligned}\quad (43)$$

Integrating throughout the volume of mercury we obtain for P, the total viscous frictional power loss

$$\begin{aligned}P &= \mu \int_v \left\{ 2 \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] \right. \\ &\quad \left. + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)^2 \right\} dv.\end{aligned}\quad (44)$$

Holding the shape of the velocity field fixed let us give V_R a variation δV_R , the corresponding variations in v_x , v_y , and v_z being δv_x , δv_y , and δv_z , respectively. At every point δV_R thus has the same direction as V_R ; hence it is really

$|V_R|$ which is being changed. We require also that the varied field satisfy the actual surface boundary condition that at the solid boundary surface provided by the channel the fluid velocity be that of the solid surface. We thus have at the mercury surface

V_R = relative velocity of solid bounding surface at such a surface,

$$\delta V_R = 0 \text{ at a solid bounding surface,} \quad (45)$$

Shear stress tangent to a free surface = 0
for the actual velocity field.

The variation in P can now be obtained from Eq. 44, thus

$$\begin{aligned} \delta P = 2\mu \int_V \left\{ 2 \left[\frac{\partial v_x}{\partial x} \delta \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \delta \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \delta \frac{\partial v_z}{\partial z} \right] \right. \\ + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \left(\delta \frac{\partial v_x}{\partial y} + \delta \frac{\partial v_y}{\partial x} \right) \\ + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \left(\delta \frac{\partial v_y}{\partial z} + \delta \frac{\partial v_z}{\partial y} \right) \\ \left. + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \left(\delta \frac{\partial v_z}{\partial x} + \delta \frac{\partial v_x}{\partial z} \right) \right\} dv. \end{aligned} \quad (46)$$

Since

$$\delta \frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x} \delta v_x, \quad \delta \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial y} \delta v_y, \quad \text{etc.} \quad (47)$$

it follows that Eq. 46 may be written

$$\begin{aligned} \delta P = 2\mu \int_V \left\{ 2 \left[\frac{\partial v_x}{\partial x} \frac{\partial}{\partial x} \delta v_x + \frac{\partial v_y}{\partial y} \frac{\partial}{\partial y} \delta v_y + \frac{\partial v_z}{\partial z} \frac{\partial}{\partial z} \delta v_z \right] \right. \\ + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \left(\frac{\partial}{\partial y} \delta v_x + \frac{\partial}{\partial x} \delta v_y \right) \\ + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \left(\frac{\partial}{\partial z} \delta v_y + \frac{\partial}{\partial y} \delta v_z \right) \\ \left. + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \left(\frac{\partial}{\partial x} \delta v_z + \frac{\partial}{\partial z} \delta v_x \right) \right\} dv. \end{aligned} \quad (48)$$

Considering only the first term in the integrand of this integral we have

$$4\mu \int_y \int_z \int_x \frac{\partial v_x}{\partial x} \frac{\partial}{\partial x} \delta v_x dydzdx. \quad (49)$$

Integrating by parts this becomes

$$4\mu \left\{ \int_y \int_z \left[\frac{\partial v_x}{\partial x} \delta v_x \right]_{x_1(y,z)}^{x_2(y,z)} dydz - \int_y \int_z \int_x \frac{\partial^2 v_x}{\partial x^2} \delta v_x dydzdx \right\} \quad (50)$$

where $x_1(y,z)$ and $x_2(y,z)$ are values of x on the surface. There may be more than two such values, but this causes no difficulty. Denoting the direction cosines of the normal to the surface by $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ Eq. 50 becomes

$$4\mu \left\{ \int_a \frac{\partial v_x}{\partial x} \delta v_x \cos\alpha_1 da - \int_v \frac{\partial^2 v_x}{\partial x^2} \delta v_x dv \right\} \quad (51)$$

where the first integral extends over the surface of the mercury, and the second extends throughout its volume. Proceeding in the same manner with the other terms in the integrand of Eq. 48, Eq. 48 becomes

$$\begin{aligned} \delta P = 2\mu \int_a \left\{ 2 \left[\frac{\partial v_x}{\partial x} \delta v_x \cos\alpha_1 + \frac{\partial v_y}{\partial y} \delta v_y \cos\beta_1 + \frac{\partial v_z}{\partial z} \delta v_z \cos\gamma_1 \right] \right. \\ + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) (\delta v_x \cos\beta_1 + \delta v_y \cos\alpha_1) \\ + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) (\delta v_y \cos\gamma_1 + \delta v_z \cos\beta_1) \\ + \left. \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) (\delta v_z \cos\alpha_1 + \delta v_x \cos\gamma_1) \right\} da \\ - 2\mu \int_v \left\{ 2 \left[\frac{\partial^2 v_x}{\partial x^2} \delta v_x + \frac{\partial^2 v_y}{\partial y^2} \delta v_y + \frac{\partial^2 v_z}{\partial z^2} \delta v_z \right] \right. \\ + \left. \left(\delta v_x \frac{\partial}{\partial y} + \delta v_y \frac{\partial}{\partial x} \right) \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right. \end{aligned} \quad (52)$$

$$\begin{aligned}
& + \left(\delta v_y \frac{\partial}{\partial z} + \delta v_z \frac{\partial}{\partial y} \right) \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\
& + \left(\delta v_z \frac{\partial}{\partial x} + \delta v_x \frac{\partial}{\partial z} \right) \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \} dv.
\end{aligned}$$

In view of Eq. 41, Eq. 52 may be written

$$\begin{aligned}
\delta P = 2 \int_a [& (\sigma_x \cos \alpha_1 + \tau_{xy} \cos \beta_1 + \tau_{xz} \cos \gamma_1) \delta v_x \\
& + (\tau_{xy} \cos \alpha_1 + \sigma_y \cos \beta_1 + \tau_{yz} \cos \gamma_1) \delta v_y \\
& + (\tau_{xz} \cos \alpha_1 + \tau_{yz} \cos \beta_1 + \sigma_z \cos \gamma_1) \delta v_z] da \\
& - 2\mu \int_v \left\{ \left[2 \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \right] \delta v_x \right. \\
& + \left[2 \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) + \frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] \delta v_y \\
& + \left. \left[2 \frac{\partial^2 v_z}{\partial z^2} + \frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \right] \delta v_z \right\} dv. \quad (53)
\end{aligned}$$

But the coefficients of δv_x , δv_y , and δv_z in the surface integral are the components of surface stress; and in the volume integral we note that since the fluid is incompressible

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \quad (54)$$

whence

$$\begin{aligned}
\frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) &= - \frac{\partial^2 v_x}{\partial x^2}, \\
\frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial z} + \frac{\partial v_x}{\partial x} \right) &= - \frac{\partial^2 v_y}{\partial y^2}, \\
\frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) &= - \frac{\partial^2 v_z}{\partial z^2}.
\end{aligned} \quad (55)$$

It follows that Eq. 53 may be written

$$\begin{aligned} \delta P = 2 \int_a S \cdot \delta V_R da - 2\mu \int_v \left[\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \delta v_x \right. \\ \left. + \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \delta v_y \right. \\ \left. + \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \delta v_z \right] dv \end{aligned} \quad (56)$$

where S is the surface stress vector. Equation 56 may be written

$$\delta P = 2 \int_a S \cdot \delta V_R da - 2\mu \int_v (\nabla^2 V_R) \cdot \delta V_R dv. \quad (57)$$

But δV_R vanishes at the channel walls, and the tangential component of S vanishes on the free surface, which causes $S \cdot \delta V_R$ to vanish there also. It follows that the integrand of the surface integral vanishes over the entire mercury surface; hence this integral vanishes, and Eq. 57 reduces to

$$\delta P = - 2\mu \int_v (\nabla^2 V_R) \cdot \delta V_R dv. \quad (58)$$

The condition for dynamic equilibrium in the liquid is

$$\rho V_R \cdot \nabla V_R = - \nabla p - \rho \nabla \Phi - 2\rho \vec{\beta} \times V_R + \mu \nabla^2 V_R \quad (59)$$

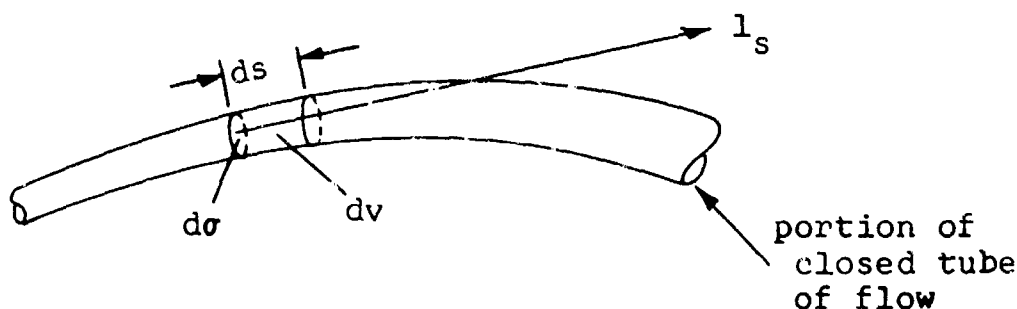
where ρ is the density of mercury, and p is the pressure. Here the left-hand side is the mass times acceleration per unit volume, and the terms on the right-hand side are the force contributions per unit volume due to the fluid pressure, the centrifugal force field (Eq. 7), the Coriolis force field (Eq. 8), and the viscous frictional force, respectively. Multiplying Eq. 59 by δV_R and integrating throughout the volume of the mercury we obtain

$$\begin{aligned} \mu \int_v (\nabla^2 V_R) \cdot \delta V_R dv = \int_v (\rho V_R \cdot \nabla V_R + \nabla p + \rho \nabla \Phi \\ + 2 \rho \vec{\beta} \times V_R) \cdot \delta V_R dv. \end{aligned} \quad (60)$$

In order to evaluate this integral let us consider the volume to be broken up into elementary tubes of flow, corresponding to which we let the element of volume dv be

$$dv = d\sigma ds \quad (61)$$

where ds is the length and $d\sigma$ is the cross-section of an elementary tube, as shown in the following figure. Also let the volume of mercury which flows across any cross-section



of the tube per unit time in the direction of the unit tangential vector l_s , in the flow δV_R be $d\delta q$; then

$$\delta V_R = \frac{d\delta q}{d\sigma} l_s. \quad (62)$$

Multiplying by Eq. 61 now gives

$$\delta V_R dv = l_s ds d\delta q, \quad (63)$$

whence Eq. 60 becomes

$$\begin{aligned} \mu \int_V (\nabla^2 V_R) \cdot \delta V_R dv = & \int_{\text{all tubes}} d\delta q \oint_{\text{line of flow}} [\rho l_s \cdot (V_R \cdot \nabla V_R) \\ & + l_s \cdot \nabla p + \rho l_s \cdot \nabla \Phi + 2\rho l_s \cdot \vec{\beta} \times V_R] ds. \end{aligned} \quad (64)$$

But the triple product $l_s \cdot \vec{\beta} \times V_R$ vanishes since l_s and V_R have the same direction; also

$$\oint l_s \cdot \nabla p ds = \oint \frac{dp}{ds} ds = \oint dp = 0, \quad (65)$$

$$\oint \rho \mathbf{l}_s \cdot \nabla \Phi ds = \oint \rho \frac{d\Phi}{ds} ds = \rho \oint d\Phi = 0. \quad (66)$$

Substituting in Eq. 64 and noting Eq. 58, we obtain

$$\delta P = - 2 \int_{\text{all tubes}} d\delta q \oint_{\text{line of flow}} \rho \mathbf{l}_s \cdot (\mathbf{V}_R \cdot \nabla \mathbf{V}_R) ds. \quad (67)$$

But

$$\begin{aligned} \mathbf{V}_R \cdot \nabla \mathbf{V}_R &= |\mathbf{V}_R| \mathbf{l}_s \cdot \nabla \mathbf{V}_R = |\mathbf{V}_R| \frac{d\mathbf{V}_R}{ds} \\ \mathbf{V}_R \cdot \nabla \mathbf{V}_R &= |\mathbf{V}_R| \left(\frac{d|\mathbf{V}_R|}{ds} \mathbf{l}_s + \frac{|\mathbf{V}_R|}{R} \mathbf{l}_N \right) \end{aligned} \quad (68)$$

where R is the radius of curvature of the line of flow, and \mathbf{l}_N is a unit principal normal vector pointing toward the center of curvature of this line. Substituting in Eq. 67 now gives

$$\begin{aligned} \delta P &= - 2 \int_{\text{all tubes}} d\delta q \oint_{\text{line of flow}} \rho |\mathbf{V}_R| \frac{d|\mathbf{V}_R|}{ds} ds \\ &= - \rho \int_{\text{all tubes}} d\delta q \oint_{\text{line of flow}} d(|\mathbf{V}_R|^2), \quad \delta P = 0. \end{aligned} \quad (69)$$

The actual value of P is therefore stationary when the velocity field is given the variation $\delta \mathbf{V}_R$.

In order to show that this value of P is actually a minimum we consider the second-order term which was not included in Eq. 46, namely

$$\begin{aligned} \mu \int_V \left\{ 2 \left[\left(\frac{\partial \delta v_x}{\partial x} \right)^2 + \left(\frac{\partial \delta v_y}{\partial y} \right)^2 + \left(\frac{\partial \delta v_z}{\partial z} \right)^2 \right] \right. \\ \left. + \left(\frac{\partial \delta v_x}{\partial y} + \frac{\partial \delta v_y}{\partial x} \right)^2 + \left(\frac{\partial \delta v_y}{\partial z} + \frac{\partial \delta v_z}{\partial y} \right)^2 \right. \\ \left. + \left(\frac{\partial \delta v_z}{\partial x} + \frac{\partial \delta v_x}{\partial z} \right)^2 \right\} dv. \end{aligned} \quad (70)$$

This term is evidently positive, and is the viscous frictional power loss which would be obtained with the velocity distribution δV_R alone. Since the first-order terms vanish, as indicated by Eq. 69, the increase in P due to δV_R is given by the positive quantity (Eq. 70). Since this vanishes if and only if $\delta V_R \equiv 0$, we see that P takes its smallest value when $\delta V_R \equiv 0$, and hence can only be increased by deviating from the actual velocity distribution V_R in the manner described.

As an application of this principle of least power loss we see immediately that in the case of the eccentric annular shape the relative, or apparent, velocity of the mercury will in some places be greater than, and in other places smaller than that of the channel. It follows that an observer riding with the main body would see the mercury going in the direction of motion of the channel in some places, and in the opposite direction in others. This is the situation which pertains in a wave motion.

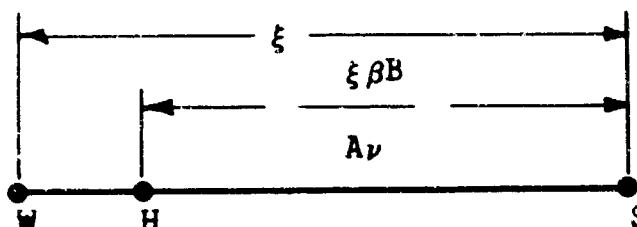
As another application we see that since P is a minimum for the actual distribution of V_R it is not sharply dependent upon this distribution. It follows that if an approximate distribution of V_R is obtained by some approximation procedure, the accuracy obtained for P is much greater than that obtained for V_R . This is important, for the quantity in which we are really interested is P , since, as we saw in Eq. 40, it is P which determines the rate of damping of the wobble. Finally we note that the approximate value of P obtained is slightly too large, since it would be reduced if the actual distribution of V_R instead of the approximate one were used in computing it.

IMPROVED TREATMENT OF THE ACTION OF THE DAMPER ON THE MAIN SPINNING BODY

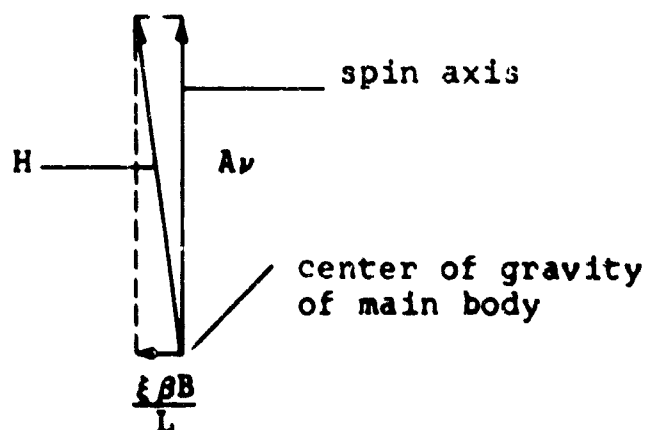
The effects of the forces exerted by the mercury on the channel, and hence on the main spinning body, were previously considered (pp. 13-14) with regard to their effect in modifying the motion of this body. The treatment was, of course, approximate; nevertheless it was adequate to show how the damper reduces the precession, or wobble, how it reduces the precessional angular velocity, and how it alters the precession axis. It is open to the objection, however, that the forces exerted by the mercury were computed using the motion which the body had prior to being acted upon by these forces. Although this situation suggests a method of successive

corrections wherein we first recompute the forces using W_1 instead of W (see figure on page 14), and the new value of precessional angular velocity; and then compute second corrections to the motion of the main body--and so on; nevertheless we shall avoid this by taking immediate account of the fact that with the damper acting the angular momentum vector H of the main spinning body does not lie along the precession axis.

Noting the figure below and the figure on the bottom of page 2 we see that for $WS = \xi$ small the H vector is, closely,



composed of a component of magnitude $A\nu$ lying along the spin axis, and a component of magnitude $\frac{\xi\beta B}{L}$ at right angles to this axis (figure below). Denoting the point where the line



extending from the center of gravity of the main body along the H vector penetrates the plane of the damper by H , which should cause no ambiguity, we see that

$$\overline{HS} = -\frac{\xi\beta B}{A\nu} L = \frac{\xi\beta B}{A\nu}. \quad (71)$$

Since the velocity of the tip of the H vector is the torque due to the external forces, in this case those exerted by the mercury, it follows, noting the figure on page 13, the fact that

$$\overline{WH} = \xi \left(1 - \frac{\beta B}{A\nu} \right), \quad (72)$$

and the fact that $\vec{\tau}_x$ and $\vec{\tau}_y$ must be multiplied by 2 in order to include the effect of the other damper, that

$$2 |\vec{\tau}_x| = -\frac{d}{dt} \left(\frac{\xi\beta B}{A\nu} \frac{A\nu}{L} \right) = -\frac{B}{L} \frac{d}{dt} (\xi\beta), \quad (73)$$

$$2 |\vec{\tau}_y| = \xi \left(1 - \frac{\beta B}{A\nu} \right) \beta \left(\frac{A\nu}{L} \right) = \frac{\xi\beta}{L} (A\nu - B\beta). \quad (74)$$

In connection with Eq. 73 we note that although W is not stationary, it is, nevertheless, the instantaneous center of S in its motion. The velocity component of S in the direction \overline{WS} is therefore zero; hence the component of the velocity of point H in this direction is the same as the component of the velocity of point H relative to S in this direction. Equation 73 therefore follows from Eq. 71.

If we let λ denote the decrease in precessional angular velocity due to the dampers, thus

$$\lambda = \frac{A\nu}{B} - \beta, \quad (75)$$

then Eq. 74 becomes

$$2 |\vec{\tau}_y| = \frac{\xi\lambda\beta B}{L}. \quad (76)$$

Substituting Eqs. 30 and 34 in Eqs. 73, 74, and 76, we obtain, respectively,

$$\frac{d}{dt} (\xi\beta) = 2 \left(\frac{ML^2}{B} \right) \beta^2 \gamma_0, \quad (77)$$

$$\xi (A\nu - B\beta) = 2ML^2 \beta x_0, \quad (78)$$

$$\xi_{AB} = 2ML^2 \beta x_0. \quad (79)$$

Further consideration of these equations must await the determination of x_0 and y_0 .

ECCENTRIC ANNULAR CONFIGURATION--NATURE OF THE FREE SURFACE OF THE MERCURY. SPECIFICATION OF THE SHAPE OF THE VELOCITY FIELD

We saw earlier that a given mercury damper may operate in several configurations (pp. 2-7) depending upon the size of the wobble, quantity of mercury, and dimensions of the channel; also we saw that one mercury damper may not experience the same sequence of configurations as another. However, in all cases the final configuration is the same for all mercury dampers, namely, eccentric annular. The determination of the nature of the final damping of the wobble hence in all cases requires the analysis of the eccentric annular configuration. We must, in particular, determine the viscous frictional loss P , since we saw from Eq. 40 that it is this quantity which is of primary importance in damping out the wobble. In determining P we shall apply the minimum principle derived previously (pp. 17-25) to determine the velocity distribution; however, in order to do this we must first specify the shape of the velocity field.

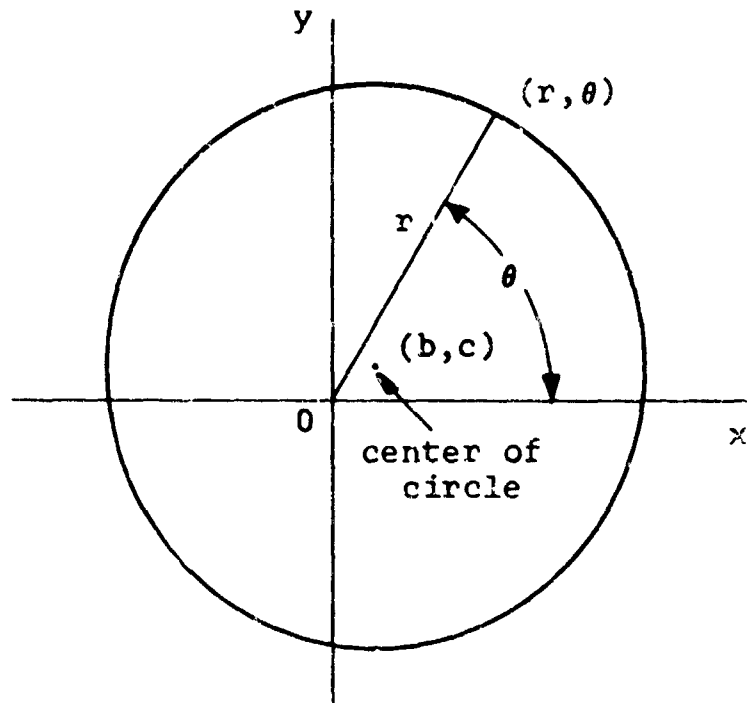
We saw on page 5 that if there were no wall friction the inner free mercury surface would be a circular cylinder centered at W as shown in the figure on page 5. With wall friction, however, this inner surface is no longer exactly a circular cylinder, and is no longer centered on W , but is displaced as indicated in the figure on page 15. Since for our purpose the circular shape offers no particular advantage, this deviation from it is no cause for concern.

Using polar coordinates r and θ let us consider the function

$$r = a + b \cos \theta + c \sin \theta \quad (80)$$

where b and c are small compared with a , as shown in the figure on the following page. Equation 80 may be written

$$r^2 = ar + bx + cy; \quad (81)$$



whence

$$\begin{aligned} r^2 - 2ar + a^2 &= -ar + a^2 + bx + cy, \\ (r-a)^2 &= -ar + a^2 + bx + cy. \end{aligned} \quad (82)$$

Retaining only first-order terms in b and c we see from Eq. 80 that $(r-a)^2$ is of second order in these quantities; hence the left-hand side of Eq. 82 may be replaced by zero, giving

$$r = a + \frac{b}{a}x + \frac{c}{a}y. \quad (83)$$

Squaring and omitting terms of order higher than first order in b and c , we obtain

$$x^2 + y^2 = a^2 + 2bx + 2cy;$$

or, again good through first-order terms,

$$(x-b)^2 + (y-c)^2 = a^2. \quad (84)$$

We thus see that good through first-order terms in b and c the polar curve (Eq. 80) is a circle of radius a whose center is at point $x = b, y = c$. To this accuracy Eq. 80 may be

used in place of the equation for a circle, and for our purpose is much more convenient.

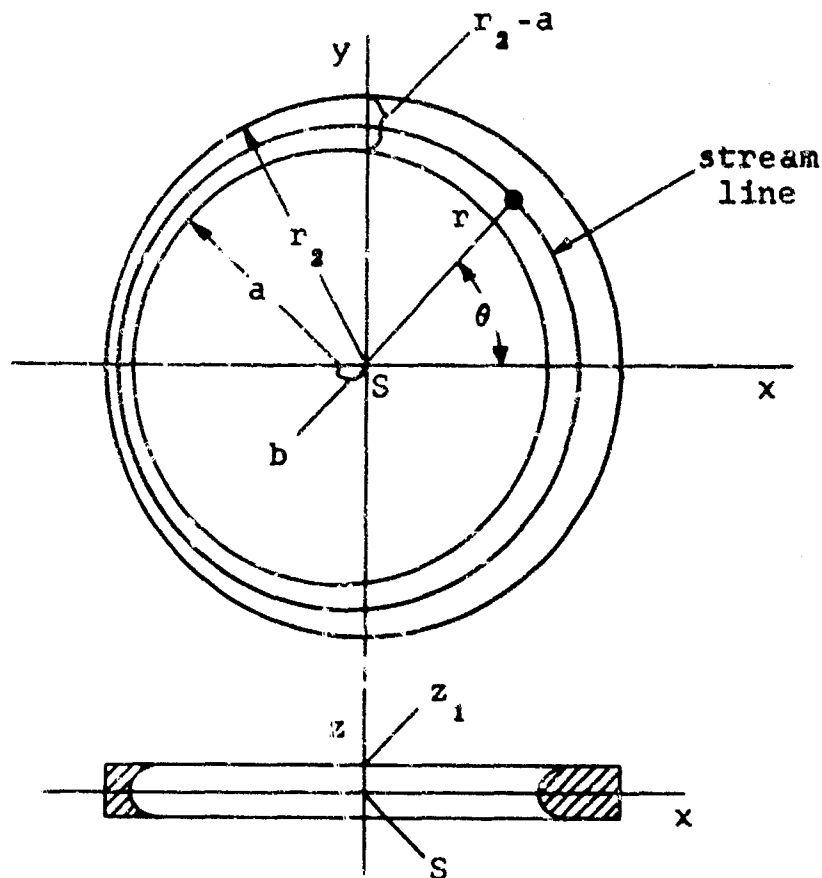
For the present we shall confine our attention to the case where the eccentricity (distance of center from S) of the inner mercury surface is small, which corresponds to the final phase in the operation of any mercury damper. It is evident that any particle of mercury which is in contact with the channel wall moves in a circle of zero eccentricity, since it must have the same velocity as that point on the wall with which it is in contact. Supposing as an approximation that all of the particles of mercury move in circular paths it follows that the eccentricities of these paths increase from zero as we move away from the channel walls and into the mercury. In particular we see that the central part of the free surface of the mercury bulges outward so as to increase the depth of the mercury where it is deepest (radially), and shrinks inward so as to decrease this depth where it is most shallow. The free surface is hence by no means cylindrical. Keeping these facts in mind and noting Eq. 80 we shall, as an approximation, choose the function

$$r = r_2 - \lambda(r_2 - a) - \lambda^2 b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \quad (85)$$

to specify the shape of the velocity field. Here cylindrical coordinates are used, the origin being at S, the center of the channel (as shown in figure on the following page); also

$$\begin{aligned} r_2 &= \text{outer radius of channel,} \\ z_1 &= \text{half the axial (vertical) width of the} \\ &\quad \text{channel,} \\ a &= \text{radius of inner free surface with no} \\ &\quad \text{wobble,} \\ b &= \text{worst eccentricity,} \\ \lambda &= \text{parameter.} \end{aligned} \quad (86)$$

For any values of λ , b , and z Eq. 85 gives, accurate through first-order terms in b , a horizontal circle. As the parameter λ varies from zero to one with b and z fixed a family of circles is obtained, the first being on the outer channel wall, and the last lying on the free surface. When $z = \pm z_1$, or $\lambda = 0$, corresponding to channel walls, the eccentricity of these circles is zero. When $\lambda = 1$ Eq. 85 becomes the equation of the free surface, shown in the figure on the following page. The circle which lies at the center of the free surface, for which $z = 0$, $\lambda = 1$, has the greatest eccentricity, namely b .



The total volume of mercury is

$$\begin{aligned}
 \text{Volume of mercury} &= \int_{-z_1}^{z_1} \int_0^{2\pi} \int_{a-b}^{r_2} r dr d\theta dz \\
 &= \frac{1}{2} \int_{-z_1}^{z_1} \int_0^{2\pi} \left\{ r_2^2 - a^2 + 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right. \\
 &\quad \left. - b^2 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2 \cos^2 \theta \right\} dz d\theta \\
 &= \pi \int_{-z_1}^{z_1} \left\{ r_2^2 - a^2 - \frac{b^2}{2} \left[1 - 2 \left(\frac{z}{z_1} \right)^2 + \left(\frac{z}{z_1} \right)^4 \right] \right\} dz \\
 &= 2\pi z_1 (r_2^2 - a^2) - \pi b^2 z_1 \left(1 - \frac{2}{3} + \frac{1}{5} \right)
 \end{aligned} \tag{87}$$

$$\text{Volume of mercury} = 2\pi z_1 \left[r_2^2 - a^2 - \frac{4}{15} b^2 \right]. \quad (88)$$

Accurate through first-order terms in b we thus have

$$\text{Volume of mercury} \approx 2\pi z_1 (r_2^2 - a^2). \quad (89)$$

From Eq. 89 we see that accurate through first-order terms in b the radius a of the circles which compose the free surface of the mercury is the same as that of the inner free cylindrical surface of the mercury when there is no wobble.

Referring to the figure on the previous page we see that the y and z coordinates of the center of gravity of the mercury are both zero due to symmetry. The x coordinate \bar{x} times the mercury volume is

$$\begin{aligned} \bar{x} \text{ times mercury volume} &= \int_{-z_1}^{z_1} \int_0^{2\pi} \int_{a-b}^{r_2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta r^2 \cos \theta \, dz d\theta dr \\ &= \frac{1}{3} \int_{-z_1}^{z_1} \int_0^{2\pi} \left\{ r_2^3 - a^3 + 3a^2 b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right. \\ &\quad \left. - 3ab^2 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2 \cos^2 \theta \right. \\ &\quad \left. + b^3 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2 \cos^3 \theta \right\} \cos \theta \, dz d\theta \\ &= \frac{2\pi}{3} \int_{-z_1}^{z_1} \left\{ \frac{3}{2} a^2 b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] + \frac{3}{8} b^3 \left[1 - 5 \left(\frac{z}{z_1} \right)^2 \right. \right. \\ &\quad \left. \left. + 3 \left(\frac{z}{z_1} \right)^4 - \left(\frac{z}{z_1} \right)^6 \right] \right\} dz \\ &= 2\pi b z_1 \left[a^2 \left(1 - \frac{1}{3} \right) + \frac{b^2}{4} \left(1 - \frac{3}{3} + \frac{3}{5} - \frac{1}{7} \right) \right] \end{aligned} \quad (90)$$

$$\bar{x} \text{ times mercury volume} = 2\pi b z_1 \left(\frac{2}{3} a^2 + \frac{4}{35} b^2 \right). \quad (91)$$

Dividing by the mercury volume (Eq. 88) we now obtain

$$\bar{x} = \frac{2b(35a^2 + 6b^2)}{7[15(r_s^2 - a^2) - 4b^2]} \quad (92)$$

Accurate through first-order terms in b we thus have

$$\bar{x} \approx \frac{2ba^2}{3(r_s^2 - a^2)} \quad (93)$$

The radius of curvature of the curve

$$r = f(\theta) \quad (94)$$

in polar coordinates is

$$R = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'r'' - rr'^2} \quad (95)$$

where a prime indicates differentiation with respect to θ . Applying this to the function

$$r = a_0 + b_0 \cos \theta \quad (96)$$

where a_0 and b_0 are constants we obtain

$$R = \frac{(a_0^2 + 2a_0b_0 \cos \theta + b_0^2 \cos^2 \theta + b_0^2 \sin^2 \theta)^{\frac{3}{2}}}{a_0^2 + 2a_0b_0 \cos \theta + b_0^2 \cos^2 \theta + 2b_0^2 \sin^2 \theta + a_0b_0 \cos \theta + b_0^2 \cos^2 \theta}$$

$$R = \frac{(a_0^2 + b_0^2 + 2a_0b_0 \cos \theta)^{\frac{3}{2}}}{a_0^2 + 3a_0b_0 \cos \theta + 2b_0^2} \quad (97)$$

Applying the binomial theorem now gives

$$R = [(a_0^2)^{\frac{3}{2}} + \frac{3}{2} (a_0^2)^{\frac{1}{2}} (b_0^2 + 2a_0b_0 \cos \theta) + \frac{3}{2} \frac{1}{2} (a_0^2)^{-\frac{1}{2}} \frac{1}{2} (b_0^2 + 2a_0b_0 \cos \theta)^2 + \dots]$$

$$\begin{aligned}
& \times [(a_0^2)^{-1} + (-1)(a_0^2)^{-2} (3a_0 b_0 \cos \theta + 2b_0^2) \\
& + (-1)(-2)(a_0^2)^{-3} \frac{1}{2} (3a_0 b_0 \cos \theta + 2b_0^2)^2 + \dots] \\
R = & a_0 + b_0 \left(\frac{1}{a_0^2} 3a_0^2 \cos \theta - a_0^3 \frac{1}{a_0^4} 3a_0 \cos \theta \right) \\
& + b_0^2 \left[a_0^3 \left(-\frac{2}{a_0^4} + (-1)(-2) \frac{1}{a_0^6} \frac{1}{2} 9a_0^2 \cos^2 \theta \right) \right. \quad (98) \\
& + \frac{3}{2} a_0 2a_0 \cos \theta \left(-\frac{1}{a_0^4} \right) 3a_0 \cos \theta \\
& \left. + \left(\frac{3}{2} a_0 + \frac{3}{2} \frac{1}{2} \frac{1}{a_0} \frac{1}{2} 4a_0^2 \cos^2 \theta \right) \frac{1}{a_0^2} \right] + \dots \\
R = & a_0 + \frac{b_0^2}{a_0} \left[-2 + \frac{3}{2} + \left(9 - 9 + \frac{3}{2} \right) \cos^2 \theta \right] + \dots \\
R = & a_0 + \frac{b_0^2}{2a_0} (3\cos^2 \theta - 1) + \dots
\end{aligned}$$

Accurate through first-order terms in b_0 we thus have

$$R \cong a_0. \quad (99)$$

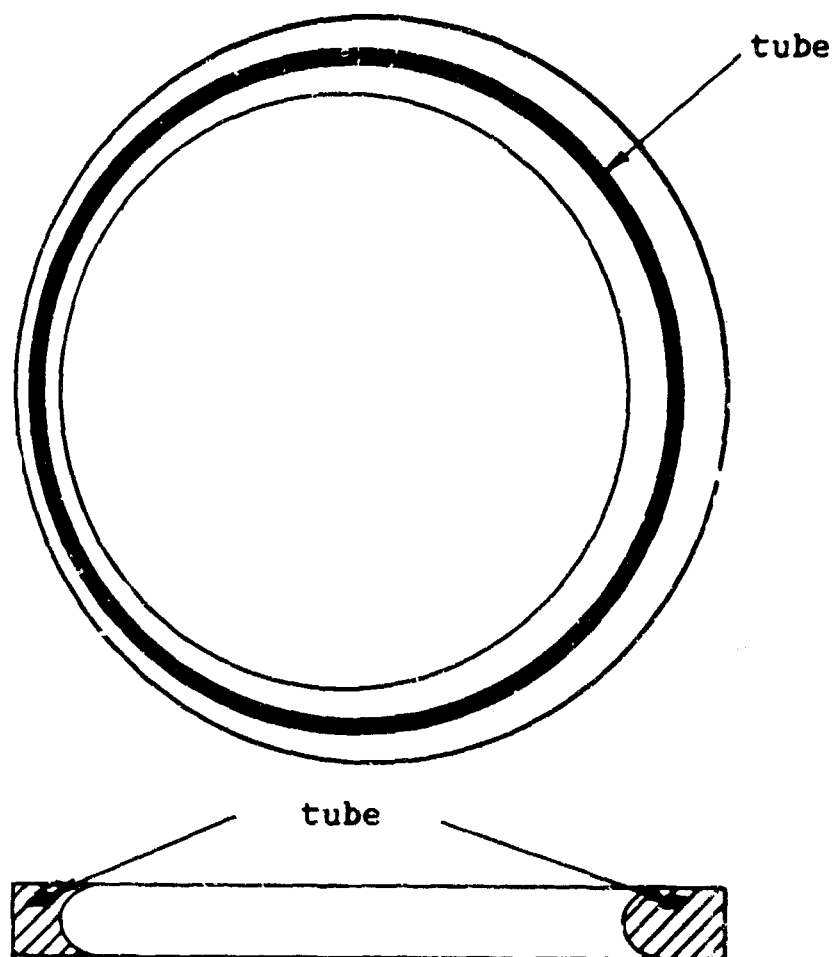
Comparing Eqs. 85 and 96 we can apply this result to Eq. 85, and thereby obtain

$$R \cong r_s - \lambda(r_s - a) \quad (100)$$

accurate through first-order terms in b .

ECCENTRIC ANNULAR CONFIGURATION--MAGNITUDE OF THE VELOCITY FIELD

Corresponding to $d\lambda dz$ we obtain a tube of flow whose cross section is an elementary parallelogram. In this tube the ranges of values of λ and z are infinitesimal; however, θ varies from zero to 2π , and the shape of the parallelogram



varies with θ , as shown in the above figure. Since the shape of the velocity field is completely specified by Eq. 85; the volume of mercury which flows in any tube $d\lambda dz$ per second can be specified by the velocity function $q(\lambda, z)$, thus

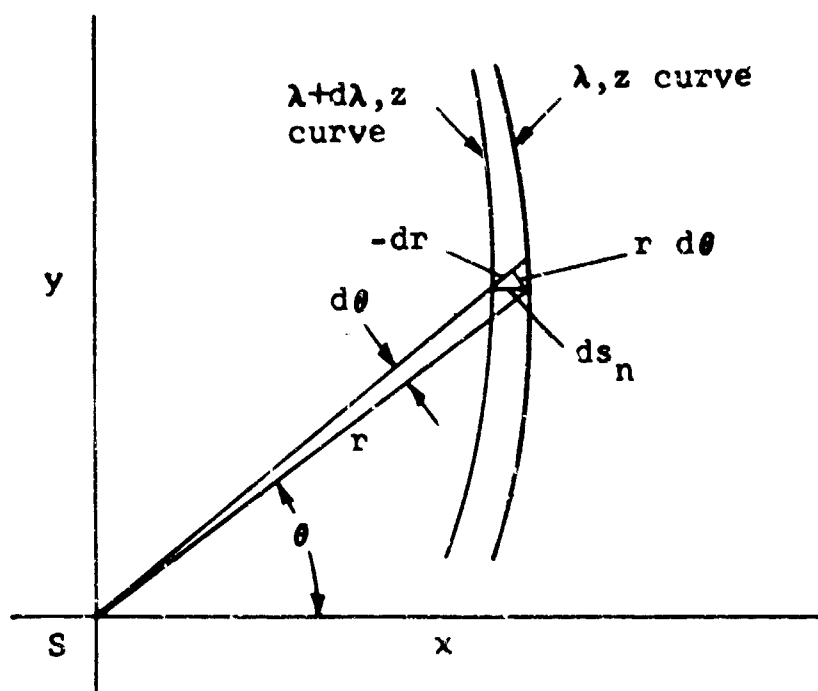
$$\text{Volume per second in tube } d\lambda dz = q(\lambda, z)d\lambda dz. \quad (101)$$

Since this volume per second is the same in all parts of the tube, the velocity at any point can be obtained by dividing Eq. 101 by the tube cross section. Accordingly we shall now determine this cross section.

The altitude of the elementary parallelogram which composes this cross section is dz . Writing Eq. 85 in the form

$$r = f(\theta, \lambda, z) \quad (102)$$

we see, looking down from the top, that the base of the parallelogram is the distance between the curves (Eq. 102) corresponding to λ, z and $\lambda + d\lambda, z$, respectively. This distance depends upon θ , and is measured along an orthogonal trajectory of the curves, as indicated in the schematic



diagram, above. The values of dr and $d\theta$ corresponding to this distance are given by the relations

$$dr = \frac{\partial r}{\partial \theta} d\theta + \frac{\partial r}{\partial \lambda} d\lambda, \quad (103)$$

$$-\frac{1}{r} \frac{dr}{d\theta} = -\frac{r}{\frac{\partial r}{\partial \theta}}, \quad (104)$$

in which r is given by Eq. 102. Here Eq. 103 is obtained by differentiating Eq. 102, and Eq. 104 is the condition for orthogonality. These equations may be written

$$\begin{aligned} dr - \frac{\partial r}{\partial \theta} d\theta &= \frac{\partial r}{\partial \lambda} d\lambda, \\ \frac{\partial r}{\partial \theta} dr + r^2 d\theta &= 0, \end{aligned} \quad (105)$$

which constitute a set of two linear, algebraic equations in dr and $d\theta$. Solving, we obtain

$$dr = \frac{r^2 \frac{\partial r}{\partial \lambda}}{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} d\lambda,$$

$$d\theta = - \frac{\frac{\partial r}{\partial \theta} \frac{\partial r}{\partial \lambda}}{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} d\lambda.$$
(106)

It follows that ds_n , the base of the parallelogram, is given by

$$ds_n = \sqrt{(dr)^2 + (rd\theta)^2} = \frac{r \left| \frac{\partial r}{\partial \lambda} \right|}{\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}} d\lambda.$$

$$ds_n = - \frac{\frac{\partial r}{\partial \lambda}}{\sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}} d\lambda,$$
(107)

since we see from Eq. 85 that $\frac{\partial r}{\partial \lambda}$ is negative.

Dividing Eq. 101 by $ds_n dz$, the cross section of the tube, using Eq. 107, we obtain for the velocity of the mercury in the tube

$$v_s = - \frac{q(\lambda, z)}{\frac{\partial r}{\partial \lambda}} \sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}.$$
(108)

Here r is given by Eq. 85, and $q(\lambda, z)$ will later be determined so as to minimize the viscous frictional power loss P .

Expanding the radical in Eq. 108 by the binomial theorem gives

$$v_s = - \frac{q(\lambda, z)}{\frac{\partial r}{\partial \lambda}} \left[1 + \frac{1}{2} \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2 - \frac{1}{8} \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^4 + \dots \right].$$
(109)

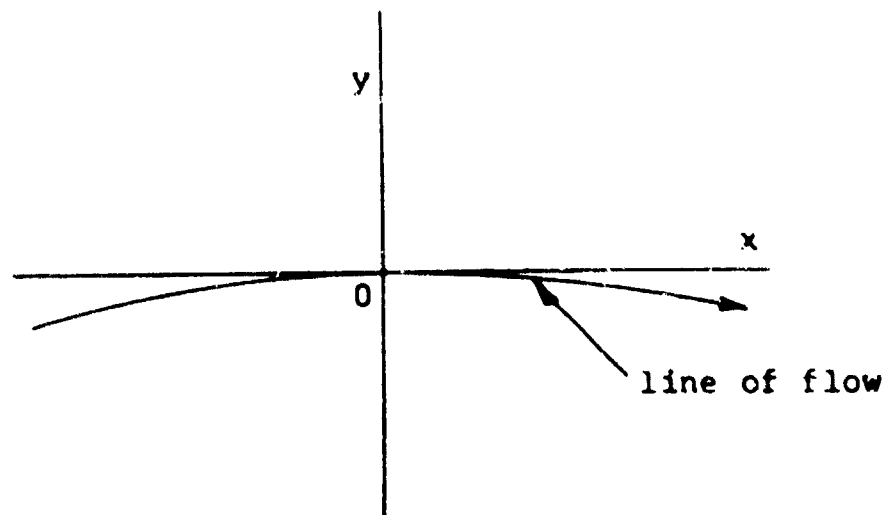
Substituting from Eq. 85 and retaining only terms through first order in b we obtain finally

$$V_s \cong \frac{q(\lambda, z)}{r_2 - a + 2\lambda b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta} \times \left[1 + \frac{1}{2} \left(\frac{\lambda^2 b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta}{r_2 - \lambda(r_2 - a) - \lambda^2 b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta} \right)^2 \right]$$

$$V_s \cong \frac{q(\lambda, z)}{r_2 - a} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}. \quad (110)$$

CALCULATION OF THE VISCOUS FRICTIONAL POWER LOSS P FOR GIVEN b AND q(λ, z)

Referring to Eq. 43 we see that in order to obtain the viscous frictional power loss per unit volume we must determine the partial derivatives of each of the velocity components V_x , V_y , and V_z with respect to x , y , and z , respectively. We may, however, place the coordinate axes in any way we choose. Let us place these with the origin at the point at which we wish to determine the loss per unit volume, the z axis vertical, and the x axis tangent to the line of flow, the positive direction being that of the flow. The positive y axis then extends outward normal to the line of flow, as shown below.



With this orientation of axes $V_z \equiv 0$; hence

$$\frac{\partial V_z}{\partial x} = 0, \quad \frac{\partial V_z}{\partial y} = 0, \quad \frac{\partial V_z}{\partial z} = 0. \quad (111)$$

Furthermore, since the fluid is incompressible

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0; \quad (112)$$

hence since the last term vanishes due to Eq. 111 it follows that

$$\frac{\partial V_y}{\partial y} = - \frac{\partial V_x}{\partial x}. \quad (113)$$

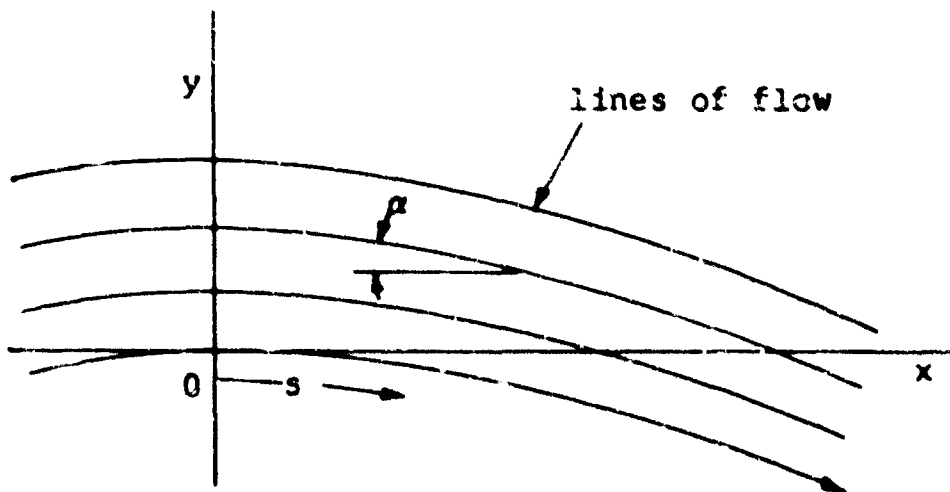
It thus remains for us to compute

$$\frac{\partial V_x}{\partial x}, \quad \frac{\partial V_x}{\partial y}, \quad \frac{\partial V_x}{\partial z}, \quad \frac{\partial V_y}{\partial x}, \quad \frac{\partial V_y}{\partial z}.$$

We shall now consider these separately.

Determination of $\frac{\partial V_x}{\partial x}$. Let s be the distance measured from the origin along a line of flow, and let α denote the angle which any arbitrary line of flow makes with the positive x direction, as shown in the figure below. Noting that V_s is a function of position we see that since

$$V_x = V_s \cos \alpha \quad (114)$$



it follows that

$$\frac{\partial V_x}{\partial x} = \frac{\partial V_s}{\partial x} \cos \alpha - V_s \sin \alpha \frac{\partial \alpha}{\partial x}. \quad (115)$$

At the origin, which is the only point in which we are interested, $\alpha = 0$; and Eq. 115 becomes

$$\frac{\partial V_x}{\partial x} = \frac{\partial V_s}{\partial x}. \quad (116)$$

Since this is merely the directional derivative of V_s in the x direction, which at the origin is the direction of increasing s , it follows that

$$\frac{\partial V_x}{\partial x} = \frac{\partial V_s}{\partial s}. \quad (117)$$

Determination of $\frac{\partial V_y}{\partial x}$. Referring to the figure on page 39 we now have

$$V_y = -V_s \sin \alpha, \quad (118)$$

$$\frac{\partial V_y}{\partial x} = -\frac{\partial V_s}{\partial x} \sin \alpha - V_s \cos \alpha \frac{\partial \alpha}{\partial x}. \quad (119)$$

At the origin $\alpha = 0$, and this becomes

$$\frac{\partial V_y}{\partial x} = -V_s \frac{\partial \alpha}{\partial x}. \quad (120)$$

But $\frac{\partial \alpha}{\partial x}$ is the directional derivative of α in the x direction, which at the origin coincides with the direction of increasing s ; hence at the origin

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial s} = \text{curvature of line of flow}. \quad (121)$$

Equation 120 thus becomes finally

$$\frac{\partial V_y}{\partial x} = -\frac{V_s}{R} \quad (122)$$

where R is the radius of curvature of the line of flow.

Determination of $\frac{\partial V_x}{\partial y}$. Differentiating Eq. 114 we obtain

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_s}{\partial y} \cos \alpha - V_s \sin \alpha \frac{\partial \alpha}{\partial y}. \quad (123)$$

At the origin $\alpha = 0$, and this becomes

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_s}{\partial y}. \quad (124)$$

But this is merely the directional derivative of V_s in the y direction, which at the origin coincides with the direction normal to the line of flow. Equation 124 thus gives

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_s}{\partial s_n} \quad (125)$$

where s_n indicates the direction of the outward normal to the line of flow.

Determination of $\frac{\partial V_x}{\partial z}$. Differentiating Eq. 114 we obtain

$$\frac{\partial V_x}{\partial z} = \frac{\partial V_s}{\partial z} \cos \alpha - V_s \sin \alpha \frac{\partial \alpha}{\partial z}. \quad (126)$$

At the origin, where $\alpha = 0$, this becomes

$$\frac{\partial V_x}{\partial z} = \frac{\partial V_s}{\partial z}. \quad (127)$$

It should be noted that the direction of the positive z axis here is the same as that of the x axis in Eq. 85. In applying Eq. 127 we may therefore use the z coordinate of Eq. 85.

Determination of $\frac{\partial V_y}{\partial z}$. Differentiating Eq. 118 we obtain

$$\frac{\partial V_y}{\partial z} = - \frac{\partial V_s}{\partial z} \sin \alpha - V_s \cos \alpha \frac{\partial \alpha}{\partial z}. \quad (128)$$

At the origin $\alpha = 0$, and this becomes

$$\frac{\partial V_y}{\partial z} = - V_s \frac{\partial \alpha}{\partial z}. \quad (129)$$

With Eq. 129, as with Eq. 127, we may use the z coordinate of Eq. 85.

Determination of $\frac{\partial V_y}{\partial y}$. Substituting Eq. 117 in Eq. 113 we obtain immediately

$$\frac{\partial V_y}{\partial y} = - \frac{\partial V_s}{\partial s}. \quad (130)$$

If, however, we differentiate Eq. 118 we obtain

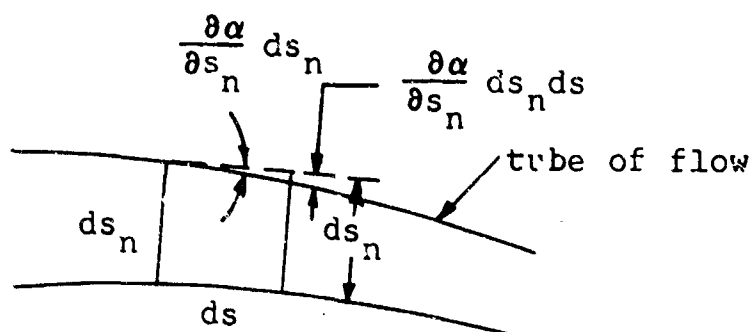
$$\frac{\partial V_y}{\partial y} = - \frac{\partial V_s}{\partial y} \sin \alpha - V_s \cos \alpha \frac{\partial \alpha}{\partial y}. \quad (131)$$

At the origin $\alpha = 0$, and the y direction is that of the outward normal; hence Eq. 131 becomes

$$\frac{\partial V_y}{\partial y} = - V_s \frac{\partial \alpha}{\partial s_n}. \quad (132)$$

That Eq. 132, which we have just obtained by differentiation, is compatible with Eq. 130, which was obtained as a consequence of the incompressibility of the fluid, can be seen by examining an elementary length of a tube of flow (see figure below). Fluid enters the left face, of area $ds_n dz$, with a velocity V_s , and leaves the right face, of area

$$(ds_n - \frac{\partial \alpha}{\partial s_n} ds_n ds) dz,$$



with a velocity $(V_s + dV_s)$. Since fluid leaves the elementary volume at the same rate at which it enters, it follows that

$$V_s ds_n dz = (V_s + dV_s) \left(1 - \frac{\partial \alpha}{\partial s_n} ds \right) ds_n dz$$

$$0 = dV_s - V_s \frac{\partial \alpha}{\partial s_n} ds, \quad (133)$$

wherein the higher order term has been omitted. Equation 133 may be written

$$\frac{\partial V_s}{\partial s} = V_s \frac{\partial \alpha}{\partial s_n}, \quad (134)$$

which equation expresses the identity of Eqs. 130 and 132. Equation 130 is easier to apply than Eq. 132, since its negative is already available as $\frac{\partial V}{\partial x}$.

Substituting Eqs. 117, 122, 125, 127, 129, and 130 in Eq. 43, we obtain

$$\text{Power loss per unit volume} = \mu \left[4 \left(\frac{\partial V_s}{\partial s} \right)^2 + \left(\frac{\partial V_s}{\partial s_n} - \frac{V_s}{R} \right)^2 + \left(\frac{\partial V_s}{\partial z} \right)^2 + \left(V_s \frac{\partial \alpha}{\partial z} \right)^2 \right]. \quad (135)$$

We shall now substitute Eq. 110 in this expression, and thereby determine the power loss per unit volume good through first-order terms in b .

Along a line of flow λ and z are constant; hence

$$\frac{\partial V_s}{\partial s} = \frac{\partial V_s}{\partial \theta} \frac{\partial \theta}{\partial s}. \quad (136)$$

But, noting Eq. 110,

$$\frac{\partial V_s}{\partial \theta} \approx \frac{2q(\lambda, z)\lambda b}{(r_2 - a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta; \quad (137)$$

also, noting Eq. 85 and applying the binomial theorem,

$$-\frac{\partial \theta}{\partial s} = \frac{1}{\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}} \approx \frac{1}{r}$$

$$= \left\{ r_2 - \lambda(r_2 - a) - \lambda^2 b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\}^{-1} \quad (138)$$

$$\frac{\partial \theta}{\partial s} \approx -\frac{1}{r_2 - \lambda(r_2 - a)} \left\{ 1 + \frac{\lambda^2 b}{r_2 - \lambda(r_2 - a)} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta + \dots \right\}.$$

Equations 137 and 138 in Eq. 136 now give

$$\frac{\partial V_s}{\partial s} \approx -\frac{2q(\lambda, z)\lambda b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \sin \theta}{(r_2 - a)^2 [r_2 - \lambda(r_2 - a)]}. \quad (139)$$

Next let us determine $\frac{\partial V_s}{\partial s_n}$. We have for fixed z

$$dV_s = \frac{\partial V_s}{\partial \lambda} d\lambda + \frac{\partial V_s}{\partial \theta} d\theta. \quad (140)$$

In the normal direction $d\theta$ is given in terms of $d\lambda$ by Eq. 106. Substituting in Eq. 140 and dividing by $-ds_n$,⁴ given by Eq. 107, we obtain

$$\frac{\partial V_s}{\partial s_n} = \frac{\frac{\partial V_s}{\partial \lambda} - \frac{\frac{\partial V_s}{\partial \theta} \frac{\partial r}{\partial \theta} \frac{\partial r}{\partial \lambda}}{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}}{\frac{\frac{\partial r}{\partial \lambda}}{\sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}}}$$

⁴ Here the minus sign is due to the fact that $\frac{\partial V_s}{\partial s_n}$ is the directional derivative of V_s in the outward direction normal to a stream line.

$$\frac{\partial V_s}{\partial s_n} = \frac{\frac{\partial V_s}{\partial \lambda}}{\frac{\partial r}{\partial \lambda}} \sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2} - \frac{\frac{1}{r^2} \frac{\partial V_s}{\partial \theta} \frac{\partial r}{\partial \theta}}{\sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}} \quad (141)$$

Substituting Eqs. 85 and 110 in Eq. 141 we obtain

$$\begin{aligned} \frac{\partial V_s}{\partial s_n} &\cong - \frac{\frac{\partial q(\lambda, z)}{\partial \lambda} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\}}{(r_2 - a) \left\{ r_2 - a + 2\lambda b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\}} \\ &\quad + \frac{2q(\lambda, z) b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta}{(r_2 - a)^2 \left\{ r_2 - a + 2\lambda b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\}} \\ \frac{\partial V_s}{\partial s_n} &\cong - \frac{\frac{\partial q(\lambda, z)}{\partial \lambda}}{(r_2 - a)^2} \left\{ 1 - \frac{4\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\} \\ &\quad + \frac{2q(\lambda, z) b}{(r_2 - a)^3} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta. \end{aligned} \quad (142)$$

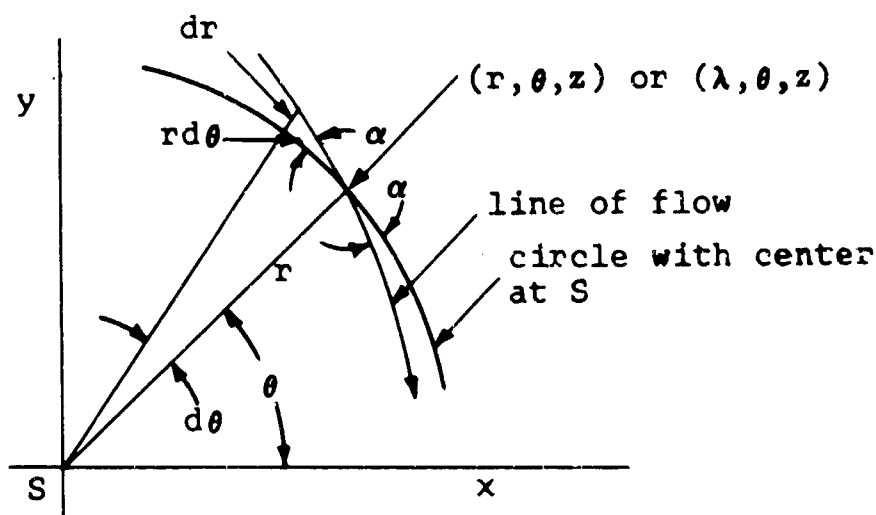
Continuing we next use Eqs. 100 and 110 to obtain

$$\frac{V_s}{R} \cong \frac{q(\lambda, z)}{(r_2 - a) [r_2 - \lambda(r_2 - a)]} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\}. \quad (143)$$

Next we obtain from Eq. 110

$$\begin{aligned} \frac{\partial V_s}{\partial z} &\cong \frac{\frac{\partial q(\lambda, z)}{\partial z}}{r_2 - a} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\} \\ &\quad + \frac{4q(\lambda, z) \lambda b z}{z_1^2 (r_2 - a)^2} \cos \theta. \end{aligned} \quad (144)$$

Lastly we shall determine $\frac{\partial \alpha}{\partial z}$. Noting the figure on the following page we see that



$$\alpha = \tan^{-1} \left(\frac{1}{r} \frac{\partial r}{\partial \theta} \right), \quad (145)$$

$$\frac{\partial \alpha}{\partial z} = \frac{r \frac{\partial^2 r}{\partial z \partial \theta} - \frac{\partial r}{\partial \theta} \frac{\partial r}{\partial z}}{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2}. \quad (146)$$

Noting Eq. 85 we see that the first and second terms in the numerator are of first and second order in b , respectively; and that the first and second terms in the denominator are of zeroth and second order in b , respectively. It follows that

$$\frac{\partial \alpha}{\partial z} \approx \frac{1}{r} \frac{\partial^2 r}{\partial z \partial \theta}, \quad (147)$$

or, substituting from Eq. 85,

$$\frac{\partial \psi}{\partial z} \approx - \frac{2b\lambda^2 z}{z_1^2 [r_2 - \lambda(r_2 - a)]} \sin \theta. \quad (148)$$

This with Eq. 110 now gives finally

$$V_s \frac{\partial \alpha}{\partial z} \approx - \frac{2q(\lambda, z) b \lambda^2 z}{z_1^2 (r_1 - a) [r_1 - \lambda (r_1 - a)]} \sin \theta. \quad (149)$$

We have now determined all of the quantities which appear in Eq. 135, the power loss per unit volume.

In order to obtain the total power loss P we must integrate Eq. 135 throughout the volume of the mercury. Using the x, y, z axes of the figure on page 31 we have

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \quad (150)$$

where r is given by Eq. 85. The Jacobian of x, y, z with respect to λ, θ, z is

$$\frac{\partial(x, y, z)}{\partial(\lambda, \theta, z)} = \begin{vmatrix} \frac{\partial r}{\partial \lambda} \cos \theta & \left(\frac{\partial r}{\partial \theta} \cos \theta - r \sin \theta \right) & \frac{\partial r}{\partial z} \cos \theta \\ \frac{\partial r}{\partial \lambda} \sin \theta & \left(\frac{\partial r}{\partial \theta} \sin \theta + r \cos \theta \right) & \frac{\partial r}{\partial z} \sin \theta \\ 0 & 0 & 1 \end{vmatrix}. \quad (151)$$

Developing by minors with respect to the elements of the last row, we obtain

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\lambda, \theta, z)} &= \begin{vmatrix} \frac{\partial r}{\partial \lambda} \cos \theta & \left(\frac{\partial r}{\partial \theta} \cos \theta - r \sin \theta \right) \\ \frac{\partial r}{\partial \lambda} \sin \theta & \left(\frac{\partial r}{\partial \theta} \sin \theta + r \cos \theta \right) \end{vmatrix} \\ \frac{\partial(x, y, z)}{\partial(\lambda, \theta, z)} &= r \frac{\partial r}{\partial \lambda}. \end{aligned} \quad (152)$$

The element of volume in the curvilinear coordinates λ, θ, z is therefore

$$dv = \left| \frac{\partial(x, y, z)}{\partial(\lambda, \theta, z)} \right| d\lambda d\theta dz = -r \frac{\partial r}{\partial \lambda} d\lambda d\theta dz \quad (153)$$

since $\frac{\partial r}{\partial \lambda}$ is negative, as is evident from Eq. 85. Equation 85 in Eq. 153 now gives finally

$$\begin{aligned} dv &= \left\{ (r_2 - a) [r_2 - \lambda(r_2 - a)] \right. \\ &\quad + \lambda b [2r_2 - 3\lambda(r_2 - a)] \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \\ &\quad \left. - 2\lambda^3 b^2 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2 \cos^2 \theta \right\} d\lambda d\theta dz. \end{aligned} \quad (154)$$

Now that we have obtained all of the quantities which appear in Eq. 135, and also the element of volume, we are in a position to write down the expression for the total power loss P. Substituting Eqs. 139, 142, 143, 144, and 149 in Eq. 135, noting Eq. 154, and integrating throughout the volume of the mercury, we obtain

$$\begin{aligned}
 P \cong \mu \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} & \left\{ \left[\frac{4q\lambda b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta}{(r_2 - a)^2 [r_2 - \lambda(r_2 - a)]} \right]^2 \right. \\
 & + \left[- \frac{\frac{\partial q}{\partial \lambda}}{(r_2 - a)^2} \left\{ 1 - \frac{4\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \right. \\
 & + \frac{2qb}{(r_2 - a)^3} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \\
 & - \frac{q}{(r_2 - a) [r_2 - \lambda(r_2 - a)]} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \left. \right]^2 \\
 & + \left[\frac{\frac{\partial q}{\partial z}}{r_2 - a} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} + \frac{4q\lambda b z \cos \theta}{z_1^2 (r_2 - a)^2} \right. \\
 & + \left. \left[\frac{2qb\lambda^2 z \sin \theta}{z_1^2 (r_2 - a) [r_2 - \lambda(r_2 - a)]} \right]^2 \right\} \left\{ (r_2 - a) [r_2 - \lambda(r_2 - a)] \right. \\
 & + \lambda b [2r_2 - 3\lambda(r_2 - a)] \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \left. \right\} d\lambda dz d\theta.
 \end{aligned} \tag{155}$$

Here $q = q(\lambda, z)$. If we knew q and b , Eq. 155 would give P closely; however, q is an unknown function which we shall now determine. In so doing Eq. 155 will be reduced, and the lowest orders of terms in b sorted out. Further reduction at this point, however, would be premature.

DETERMINATION OF $q(\lambda, z)$

Since $q(\lambda, z)$ depends upon b , let us expand this function in a Maclaurin series, thus

$$q(\lambda, z) = q_0(\lambda, z) + bq_1(\lambda, z) + b^2q_2(\lambda, z) + \dots \quad (156)$$

The problem of determining $q(\lambda, z)$ then becomes that of obtaining $q_0(\lambda, z)$, $q_1(\lambda, z)$, $q_2(\lambda, z)$, ..., which functions are independent of b . This we can do by minimizing the viscous frictional power loss P , given by Eq. 155, under the conditions that the mercury velocity be that of the channel walls wherever it is in contact with these walls, and that b be fixed at some arbitrary small value.

First, let us place $b = 0$, then Eqs. 156, 110, and 155 become, respectively,

$$q(\lambda, z) = q_0(\lambda, z), \quad (157)$$

$$V_s = \frac{q_0(\lambda, z)}{r_2 - a}, \quad (158)$$

$$P_0 = \mu \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} \left\{ \left[-\frac{\frac{\partial q_0}{\partial \lambda}}{(r_2 - a)^2} - \frac{q_0}{(r_2 - a)[r_2 - \lambda(r_2 - a)]} \right]^2 + \left[\frac{\frac{\partial q_0}{\partial z}}{r_2 - a} \right]^2 \right\} (r_2 - a)[r_2 - \lambda(r_2 - a)] d\lambda dz d\theta. \quad (159)$$

When $b = 0$ the mercury takes the shape of an annular ring, and rotates with the main spinning body as if it were a rigid, integral part of it, the apparent angular velocity being $(\beta - \nu)$ clockwise. Noting Eq. 85 with $b = 0$, it follows that

$$V_s = (\beta - \nu) [r_2 - \lambda(r_2 - a)]. \quad (160)$$

Substituting in Eq. 158 we now obtain

$$q_0(\lambda, z) = (\beta - \nu) (r_2 - a) [r_2 - \lambda(r_2 - a)]. \quad (161)$$

Substituting Eq. 161 in Eq. 159 we find that both brackets in the integrand of Eq. 159 vanish, and

$$P_0 = 0, \quad (162)$$

which we should expect, with the mercury rotating like a rigid body. It may be mentioned that Eq. 161 can also be obtained directly by minimizing Eq. 159 subject to the above-mentioned surface conditions, no use then being made of Eqs. 158 and 160.

Next, let us substitute Eq. 156 into Eq. 155, which can be visualized without rewriting Eq. 155; then the resulting expression for P becomes a power series in b , thus

$$P = P_0 + P_1 b + P_2 b^2 + \dots \quad (163)$$

Referring to the subscript on P_i or q_i as the order of that P_i or q_i , we see that the higher the order of P_i , the higher will be the order of the highest order q which it contains. In order to determine successively $q_1(\lambda, z)$, $q_2(\lambda, z)$, ... we make use of the following two facts.

1. $q(\lambda, z)$, and hence $q_1(\lambda, z)$, $q_2(\lambda, z)$, ... are such that P is a minimum, subject to the above-mentioned surface condition, regardless of the particular value chosen for b .

2. If b is taken to be sufficiently small, the sum of all the terms beyond the n th in Eq. 163 is negligible in comparison with the sum of the first n terms of this series, providing only that these n terms contain at least one non-vanishing term.

We have already seen how $q_0(\lambda, z)$ can be obtained by using but one term ($n=1$) of Eq. 163 in connection with the minimum principle.

In order to determine $q_1(\lambda, z)$ let us place $n=2$; however, since P_0 contains only $q_0(\lambda, z)$, which has already been determined, and since $P_0 = 0$, due to Eq. 162, we shall in effect try to determine $q_1(\lambda, z)$ by minimizing P_1 , in which $q_0(\lambda, z)$ has been replaced by its known expression (Eq. 161).

Substituting Eq. 156 in Eq. 155 each large bracket in the integrand becomes a power series in b . These series, however, begin with terms which are linear in b , for those terms without b , which are also those which appear in the brackets in Eq. 159, all vanish, as we have seen. Noting the squares on

the large brackets in the integrand of Eq. 155, it is now evident that the smallest power of b which can appear in the expansion of P is two; hence

$$P_1 = 0. \quad (164)$$

In view of this we place $n=3$, and try to determine $q_1(\lambda, z)$ by minimizing P_2 .

Since the coefficient of b^2 in the series obtained by squaring a power series in b which begins with the linear term is merely the square of the coefficient of b in that linear term, we can write immediately

$$\begin{aligned} P_2 = \mu \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} & \left\{ \left[\frac{4q_0\lambda \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin\theta}{(r_2-a)^2 [r_2-\lambda(r_2-a)]} \right]^2 \right. \\ & + \left[\frac{4 \frac{\partial q_0}{\partial \lambda} \lambda}{(r_2-a)^3} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta \right. \\ & - \frac{\frac{\partial q_1}{\partial \lambda}}{(r_2-a)^2} + \frac{2q_0}{(r_2-a)^3} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta \\ & + \frac{2q_0\lambda \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta}{(r_2-a)^2 [r_2-\lambda(r_2-a)]} - \frac{q_1}{(r_2-a) [r_2-\lambda(r_2-a)]} \left. \right]^2 \\ & + \left[- \frac{2 \frac{\partial q_0}{\partial z} \lambda}{(r_2-a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta + \frac{\frac{\partial q_1}{\partial z}}{r_2-a} + \frac{4q_0\lambda z \cos\theta}{z_1^2 (r_2-a)^2} \right]^2 \\ & + \left[\frac{2q_0\lambda^2 z \sin\theta}{z_1^2 (r_2-a) [r_2-\lambda(r_2-a)]} \right]^2 \left. \right\} (r_2-a) [r_2-\lambda(r_2-a)] d\lambda dz d\theta. \end{aligned} \quad (165)$$

Noting that

$$\int_0^{2\pi} (A+B\cos\theta)^2 d\theta = \pi(2A^2+B^2) \quad (166)$$

we square the brackets in Eq. 165 and carry out the integration on θ ; then substituting Eq. 161 for q_0 we obtain

$$\begin{aligned} P_2 = & \pi\mu(\beta-\nu)^2 \int_0^1 \int_{-z_1}^{z_1} \left\{ \frac{16\lambda^2 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2}{(r_2-a)^2} \right. \\ & + 4[r_2-2\lambda(r_2-a)]^2 \frac{\left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2}{(r_2-a)^4} \\ & + \frac{16\lambda^2 z^2 [r_2-\lambda(r_2-a)]^2}{z_1^4 (r_2-a)^2} + \frac{4\lambda^4 z^2}{z_1^4} \left. \right\} (r_2-a)[r_2-\lambda(r_2-a)] d\lambda dz \\ & + 2\pi\mu \int_0^1 \int_{-z_1}^{z_1} \left\{ \left[\frac{\frac{\partial q_1}{\partial \lambda}}{(r_2-a)^2} + \frac{q_1}{(r_2-a)[r_2-\lambda(r_2-a)]} \right]^2 \right. \\ & + \left. \left[\frac{\frac{\partial q_1}{\partial z}}{r_2-a} \right]^2 \right\} (r_2-a)[r_2-\lambda(r_2-a)] d\lambda dz. \end{aligned} \quad (167)$$

Here the first integral does not contain q_1 , and hence plays no role in the minimization process. The integrand in the second integral contains two squared brackets, and is hence positive. This integral therefore takes its minimum value, zero, when

$$q_1(\lambda, z) \equiv 0, \quad (168)$$

which is consistent with the required boundary conditions, and is hence the desired function.

At first sight it may seem strange that $P_1 = 0$ and $q_1(\lambda, z) \equiv 0$; however, noting Eqs. 85, 110, and the figure on page 31, we see that if b is replaced by $(-b)$, the effect is

merely to rotate the flow pattern as a whole through 180 deg, which does not alter P or $q(\lambda, z)$. P and $q(\lambda, z)$ must therefore be even functions of b ; hence

$$\begin{aligned} P_i &= 0 \text{ if } i \text{ is odd,} \\ q_i(\lambda, z) &= 0 \text{ if } i \text{ is odd.} \end{aligned} \quad (169)$$

We note that Eqs. 164 and 168 are in accord with this result.

Finally we note that since $q_0(\lambda, z)$ provides the desired mercury velocity at the channel walls, it follows that all higher order q 's must vanish at the channel walls, thus

$$q_i(\lambda, z) = 0 \text{ when } i > 0, \text{ and } \lambda = 0 \text{ or } z = \pm z_1. \quad (170)$$

DETERMINATION OF THE VISCOUS FRICTIONAL POWER LOSS P

Returning to Eq. 167 we see that since the second integral vanishes due to Eq. 168, P_2 is given by the first alone. Carrying out the integration on z , we obtain

$$\begin{aligned} P_2 &= \pi\mu(\beta-\nu)^2 \int_0^1 \left\{ \frac{16\lambda^2 16z_1}{15(r_2-a)^3} + \frac{4[r_2-2\lambda(r_2-a)]^2 16z_1}{15(r_2-a)^4} \right. \\ &\quad \left. + \frac{16\lambda^2 2z_1 [r_2-\lambda(r_2-a)]^2}{3z_1^2 (r_2-a)^3} + \frac{4\lambda^4 2z_1}{3z_1^3} \right\} (r_2-a) [r_2-\lambda(r_2-a)] d\lambda \\ &= \frac{8}{15} \pi\mu(\beta-\nu)^2 \int_0^1 \left\{ \frac{32\lambda^2 z_1}{(r_2-a)} [r_2-\lambda(r_2-a)] \right. \\ &\quad \left. + \frac{8z_1}{(r_2-a)^3} [r_2^3 + \lambda(-r_2^2(r_2-a) - 4r_2^2(r_2-a))] \right. \\ &\quad \left. + \lambda^2(4r_2(r_2-a)^2 + 4r_2(r_2-a)^2) + \lambda^3(-4(r_2-a)^3) \right\} \\ &\quad \left. + \frac{20}{z_1(r_2-a)} [\lambda^2 r_2^3 - 3\lambda^3 r_2^2(r_2-a) + 3\lambda^4 r_2(r_2-a)^2 \right. \\ &\quad \left. - \lambda^5(r_2-a)^3] + \frac{5(r_2-a)}{z_1} [\lambda^4 r_2 - \lambda^5(r_2-a)] \right\} d\lambda \end{aligned} \quad (171)$$

$$\begin{aligned}
P_2 &= \frac{8}{15} \pi \mu (\beta - \nu)^2 \left\{ \frac{32z_1}{(r_2 - a)} \left[\frac{1}{3} r_2 - \frac{1}{4} (r_2 - a) \right] \right. \\
&\quad + \frac{8z_1}{(r_2 - a)^3} \left[r_2^3 - \frac{5}{2} r_2^2 (r_2 - a) + \frac{8}{3} r_2 (r_2 - a)^2 - (r_2 - a)^3 \right] \\
&\quad + \frac{20}{z_1 (r_2 - a)} \left[\frac{1}{3} r_2^3 - \frac{3}{4} r_2^2 (r_2 - a) + \frac{3}{5} r_2 (r_2 - a)^2 \right. \\
&\quad \left. \left. - \frac{1}{6} (r_2 - a)^3 \right] + \frac{5(r_2 - a)}{z_1} \left[\frac{1}{5} r_2 - \frac{1}{6} (r_2 - a) \right] \right\} \\
&= \frac{8}{15} \pi \mu (\beta - \nu)^2 \left\{ -16z_1 + \frac{32z_1 r_2}{(r_2 - a)} - \frac{20z_1 r_2^2}{(r_2 - a)^2} + \frac{8z_1 r_2^3}{(r_2 - a)^3} \right. \\
&\quad \left. - \frac{25}{6} \frac{(r_2 - a)^2}{z_1} + \frac{13z_1 (r_2 - a)}{z_1} - \frac{15r_2^2}{z_1} + \frac{20}{3} \frac{r_2^3}{z_1 (r_2 - a)} \right\} . \\
P_2 &= \frac{4}{45} \pi \mu (\beta - \nu)^2 \left\{ 24z_1 \left[2 \left(\frac{r_2}{r_2 - a} \right)^3 - 5 \left(\frac{r_2}{r_2 - a} \right)^2 + 8 \left(\frac{r_2}{r_2 - a} \right) - 4 \right] \right. \\
&\quad \left. + \frac{r_2^2}{z_1} \left[40 \left(\frac{r_2}{r_2 - a} \right) - 90 + 78 \left(\frac{r_2 - a}{r_2} \right) - 25 \left(\frac{r_2 - a}{r_2} \right)^2 \right] \right\} . \quad (172)
\end{aligned}$$

Knowing P_2 and the fact that $P_3 = 0$, due to Eq. 169, it follows from Eq. 163 that accurate through the term containing b^3

$$P \approx P_2 b^2. \quad (173)$$

We saw on pages 15 and 16 that this quantity is of primary importance in eliminating the wobble.

EXTENSION OF THE MINIMUM PRINCIPLE FOR THE VISCOUS FRICTIONAL POWER LOSS

The class of velocity distributions considered in deriving the principle of least viscous frictional power loss (pp. 17-25) consisted of those distributions which at each

point have the right direction, and which also satisfy the actual surface boundary condition at the solid surface, as described on page 19.

Later on in applying the minimum principle to determine the velocity distribution, the shape of the velocity field was specified by Eq. 85, which was assumed to give a sufficiently good approximation if b is properly chosen. We still have the problem of determining b .

It has been convenient to picture the mercury flow as being composed of an indefinitely large number of elementary tubes of flow, a typical tube comprising the region λ to $\lambda + d\lambda$ in λ , and z to $z + dz$ in z . We may, in fact, consider the fluid to be constrained to move in a network of such tubes. If in so doing we picture the walls of the tubes as having the ability to exert constraining pressure normal to these walls and hence normal to the tubes of flow, we arrive at a situation wherein the approximate velocity distribution obtained for any value of b and given by Eqs. 85, 110, and 156 may be considered to be the exact velocity distribution that is obtained if a suitable constraining body force \mathfrak{F} per unit volume be superimposed. The body force intensity vector \mathfrak{F} is everywhere normal to the relative velocity vector V_R . Adopting this point of view we shall henceforth refer to the approximate velocity distribution corresponding to any value of b as the "constrained velocity distribution."

We can easily extend the minimum principle given on pages 17 to 25, so that it is applicable not only to the actual velocity distribution but also to the constrained velocity distribution. In order to do this we need only to add a term \mathfrak{F} to the right-hand side of Eq. 59, which leads to an additional term

$$- \int_V \mathfrak{F} \cdot \delta V_R \, dv \quad (174)$$

on the right-hand side of Eq. 60. This term vanishes, however, since \mathfrak{F} is normal to δV_R , which is parallel to V_R ; hence Eq. 60 is not altered, and the derivation continues as before. It follows that the principle of least viscous frictional power loss applies not only to the actual velocity distribution, but also to the constrained velocity distribution.

As a consequence of this extension of the minimum principle we can dispense with the assumption that the shape of the velocity field given by Eq. 85 is sufficiently close to that

of the actual distribution to permit the application of this minimum principle, since the validity of this principle has now been established for the constrained velocity distribution itself.

PRINCIPLE OF LEAST CONSTRAINT

Since the constraining body force intensity \mathfrak{F} was introduced artificially so that with it the approximate velocity distribution becomes exact, we see that the exact solution of the fluid flow problem is characterized by the relation

$$\mathfrak{F} \equiv 0. \quad (175)$$

This will automatically be realized if

$$\int_V \mathfrak{F}^2 dv = 0, \quad (176)$$

the integration extending throughout the volume of the mercury. If the solution obtained is not exact, the integral in Eq. 176, which we shall call the "constraint," will not vanish. Nevertheless the problem of obtaining an exact solution includes that of obtaining zero constraint; and in obtaining our approximate solution we shall try to achieve this condition as closely as possible, as indicated by the condition

$$\text{Constraint} = \kappa = \int_V \mathfrak{F}^2 dv = \text{minimum}. \quad (177)$$

At present we have a family of approximate solutions characterized by the parameter b and, as we shall see, by an unknown function of λ and z . These will be determined by the above "principle of least constraint," which is expressed by Eq. 177, and by a surface analogue of this principle which applies to the surface constraining pressure at the free mercury surface. This analogue will, however, be considered later (see page 64).

DETERMINATION OF THE PRESSURE FIELD

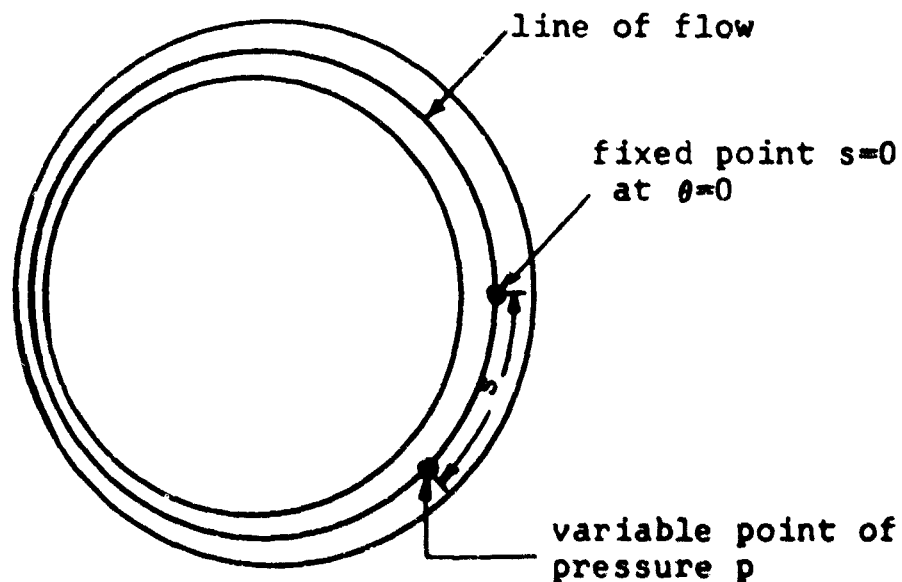
If the constraining force intensity \mathfrak{F} be included in Eq. 59 we obtain as the condition for dynamic equilibrium in the liquid

$$\rho \mathbf{V}_R \cdot \nabla \mathbf{V}_R = - \nabla p - \rho \nabla \Phi - 2\rho \vec{\beta} \times \mathbf{V}_R + \mu \nabla^2 \mathbf{V}_R + \mathfrak{F}. \quad (178)$$

We shall use this expression to determine \mathfrak{F} for use in Eq. 177; however, in order to do this we must first determine p , the pressure.

p can be partially determined by solving Eq. 178 for ∇p , multiplying by $\mathbf{l}_s ds$, and integrating along a line of flow starting at the point for which $\theta=0$, as shown in the figure below. Proceeding thus, we obtain

$$\begin{aligned} \int_0^s \mathbf{l}_s \cdot \nabla p ds = \int_0^s [& - \rho \mathbf{l}_s \cdot (\mathbf{V}_R \cdot \nabla \mathbf{V}_R) - \rho \mathbf{l}_s \cdot \nabla \Phi \\ & - 2\rho \mathbf{l}_s \cdot \vec{\beta} \times \mathbf{V}_R + \mu \mathbf{l}_s \cdot \nabla^2 \mathbf{V}_R + \mathbf{l}_s \cdot \mathfrak{F}] ds. \end{aligned} \quad (179)$$



But the triple product $\mathbf{l}_s \cdot \vec{\beta} \times \mathbf{V}_R$ vanishes since \mathbf{l}_s and \mathbf{V}_R have the same direction; also $\mathbf{l}_s \cdot \mathfrak{F}$ vanishes since \mathfrak{F} is normal to the line of flow. Finally, noting Eq. 60 and the fact that we have denoted $|\mathbf{V}_R|$ by V_s , we have

$$l_s \cdot (V_R \cdot \nabla V_R) = |V_R| \frac{d|V_R|}{ds} = \frac{1}{2} \frac{d(V_s^2)}{ds}. \quad (180)$$

In view of the above, Eq. 179 becomes

$$\int_0^s \frac{dp}{ds} ds = \int_0^s \left[-\frac{\rho}{2} \frac{d(V_s^2)}{ds} - \rho \frac{d\Phi}{ds} + \mu l_s \cdot \nabla^2 V_R \right] ds,$$

$$p = -\frac{\rho V_s^2}{2} - \rho \Phi + \mu \int_0^s l_s \cdot \nabla^2 V_R ds + \psi(\lambda, z) \quad (181)$$

wherein ψ is constant for any line of flow, but may differ from line to line. We note that a line of flow is specified by the pair of values (λ, z) which correspond to it; hence ψ is a function of λ and z .

Taking the gradient of Eq. 181 gives

$$\nabla p = -\rho V_s \nabla V_s - \rho \nabla \Phi + \mu \nabla \int_0^s l_s \cdot \nabla^2 V_R ds + \nabla \psi. \quad (182)$$

Substituting in Eq. 178 and solving for \mathfrak{F} we now obtain

$$\mathfrak{F} = \rho (V_R \cdot \nabla V_R - V_s \nabla V_s) + 2\rho \vec{\beta} \times V_R$$

$$- \mu \left(\nabla^2 V_R - \nabla \int_0^s l_s \cdot \nabla^2 V_R ds \right) + \nabla \psi. \quad (183)$$

Noting Eq. 68 and the fact that

$$\nabla V_s = \frac{\partial V_s}{\partial s} l_s + \frac{\partial V_s}{\partial s_n} l_{s_n} + \frac{\partial V_s}{\partial z} l_z; \quad (184)$$

also that

$$l_{s_n} = -l_N, \quad |V_R| = V_s; \quad (185)$$

it follows that

$$V_R \cdot \nabla V_R - V_S \nabla V_S = \left(\frac{V_S^2}{R} + V_S \frac{\partial V_S}{\partial s_n} \right) l_N - V_S \frac{\partial V_S}{\partial z} l_z. \quad (186)$$

Equation 183 thus becomes

$$\begin{aligned} \mathfrak{F} = & \rho \left(\frac{V_S^2}{R} + V_S \frac{\partial V_S}{\partial s_n} \right) l_N - \rho V_S \frac{\partial V_S}{\partial z} l_z + 2\rho \vec{\beta} \times V_R \\ & - \mu \left(\nabla^2 V_R - \nabla \int_0^s l_s \cdot \nabla^2 V_R ds \right) + \nabla \phi. \end{aligned} \quad (187)$$

Here we have seen that \mathfrak{F} is at each point normal to the line of flow. The same is true of $\nabla \phi$, since ϕ is constant along each stream line; also it is evidently true of the first three terms on the right-hand side of Eq. 187, including the Coriolis term. It follows that the one remaining term, which contains μ , is at each point normal to the line of flow.

Another fact to be noted is that since \mathfrak{F} is by definition such as to make the approximate flow physically possible, there is no question about the existence of ϕ ; hence the expression for $\nabla \phi$ obtained by solving Eq. 187 must be such that a function having it as a gradient exists. The curl of this expression must therefore vanish.

Turning to the determination of ϕ we first note that Eq. 187 may be written

$$\mathfrak{F} = U(\theta, \lambda, z) + \nabla \phi(\lambda, z) \quad (188)$$

where

$$\begin{aligned} U = & \rho \left(\frac{V_S^2}{R} + V_S \frac{\partial V_S}{\partial s_n} \right) l_N - \rho V_S \frac{\partial V_S}{\partial z} l_z + 2\rho \vec{\beta} \times V_R \\ & - \mu \left(\nabla^2 V_R - \nabla \int_0^s l_s \cdot \nabla^2 V_R ds \right). \end{aligned} \quad (189)$$

The configuration, the velocity and pressure fields, the constraining force field, and ψ also depend upon the parameter b , which, in turn, depends upon ξ , which is a measure of the angle between the spin axis and the precessor axis. Substituting Eqs. 188 and 153 in Eq. 177 we obtain for the constraint

$$\kappa = \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} (U + \nabla\psi)^2 J d\lambda dz d\theta \quad (190)$$

wherein

$$\begin{aligned} J = \left| \frac{\partial(x, y, z)}{\partial(\lambda, \theta, z)} \right| &= -r \frac{\partial r}{\partial \lambda} = \left\{ (r_2 - a) [r_2 - \lambda(r_2 - a)] \right. \\ &+ \lambda b [2r_2 - 3\lambda(r_2 - a)] \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \\ &\left. - 2\lambda^3 b^2 \left[1 - \left(\frac{z}{z_1} \right)^2 \right]^2 \cos^2 \theta \right\} \end{aligned} \quad (191)$$

due to Eqs. 153 and 154. We now wish to determine ψ so as to minimize κ .

Proceeding with ψ and κ in a manner similar to that used with q and P in the section on pages 49-53, we represent U , ψ , and κ by their Maclaurin series expansions in b , thus

$$\begin{aligned} U(\lambda, z, \theta) &= U_0(\lambda, z, \theta) + bU_1(\lambda, z, \theta) + b^2U_2(\lambda, z, \theta) + \dots, \\ \psi(\lambda, z) &= \psi_0(\lambda, z) + b\psi_1(\lambda, z) + b^2\psi_2(\lambda, z) + \dots, \\ \kappa &= \kappa_0 + b\kappa_1 + b^2\kappa_2 + \dots \end{aligned} \quad (192)$$

Substituting in Eq. 190 we obtain

$$\begin{aligned} \kappa &= \kappa_0 + b\kappa_1 + b^2\kappa_2 + \dots = \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} [U_0 + \nabla\psi_0 \\ &+ (U_1 + \nabla\psi_1)b + (U_2 + \nabla\psi_2)b^2 + \dots]^2 J d\lambda dz d\theta \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \int_{-z_1}^{z_1} \int_0^{2\pi} \{ (u_0 + \nabla\phi_0)^2 + 2(u_0 + \nabla\phi_0) \\
&\quad \cdot (u_1 + \nabla\phi_1)b + [(u_1 + \nabla\phi_1)^2 + 2(u_0 + \nabla\phi_0) \\
&\quad \cdot (u_2 + \nabla\phi_2)]b^2 + \dots \} J d\lambda dz d\theta. \quad (193)
\end{aligned}$$

b is evidently an odd function of ξ , and vanishes when $\xi=0$, in which case the spin axis and the precession axis coincide. For any value of b , and its corresponding value of ξ , ϕ , and hence $\phi_0, \phi_1, \phi_2, \dots$, are determined so that the constraint κ is minimized.

First let us place $b=0$, corresponding to which the mercury and the main body rotate together as one rigid body. Then the second and last terms in Eq. 189 vanish, since V_s does not vary in the z direction, and viscosity plays no role. Also, noting Eqs. 85, 142, 160, and 161, we have

$$\begin{aligned}
u_0 &= \rho \{ (\beta - \nu)^2 [r_s - \lambda(r_s - a)] + (\beta - \nu)^2 [r_s - \lambda(r_s - a)] \} l_N \\
&\quad - 2\rho\beta(\beta - \nu) [r_s - \lambda(r_s - a)] l_N \\
u_0 &= -2\rho\nu(\beta - \nu) [r_s - \lambda(r_s - a)] l_N \quad (194)
\end{aligned}$$

where l_N is a unit vector directed toward the axis. Placing $b=0$ in Eq. 193 we lose all but the first term in the brace; and this can be made zero by putting

$$-\nabla\phi_0 = u_0, \quad (195)$$

thus

$$-\nabla\phi_0 = \frac{\frac{d\phi_0}{d\lambda}}{\frac{dr}{d\lambda}} l_N = -2\rho\nu(\beta - \nu) [r_s - \lambda(r_s - a)] l_N, \quad (196)$$

$$\begin{aligned}
\frac{d\phi_0}{d\lambda} &= 2\rho\nu(\beta - \nu)(r_s - a) [r_s - \lambda(r_s - a)], \\
\phi_0 &= -\rho\nu(\beta - \nu) [r_s - \lambda(r_s - a)]^2 + \rho\nu(\beta - \nu)a^2 + c_0 \quad (197)
\end{aligned}$$

where c_0 is an integration constant. Here the second term is added artificially so that

$$\psi_0 = c_0 \quad \text{when} \quad \lambda = 1, \quad (198)$$

which will be convenient later.

Next, noting Eqs. 85, 110, 156, 169, and the figure on page 31, we see that if b is replaced by $(-b)$ the effect is merely to rotate the flow pattern as a whole through 180 deg. This does not alter κ , which must therefore be an even function of b . Furthermore if we consider the pressures at corresponding points of the flow patterns for b and $(-b)$, respectively, these being points which differ only by 180 deg in θ , the values of λ and z being the same for both; we see that all terms on both sides of Eq. 181 except ψ are the same at corresponding points (see NOTE, below). It follows that ψ is the same at corresponding points. Since these points have the same values of λ and z , but values of b which differ only in sign, it follows that ψ is an even function of b . Since κ and ψ are even, the κ 's and ψ 's with odd subscripts vanish, and

$$\kappa = b^2 \kappa_2 + b^4 \kappa_4 + b^6 \kappa_6 + \dots, \quad (199)$$

$$\psi(\lambda, z) = \psi_0(\lambda, z) + b^2 \psi_2(\lambda, z) + b^4 \psi_4(\lambda, z) + \dots. \quad (200)$$

Here $\kappa_0=0$ due to Eq. 193 with $b=0$, and Eq. 195. It is now evident that good through first-order terms in b the pressure p is given by replacing ψ by Eq. 197 in Eq. 181, thus

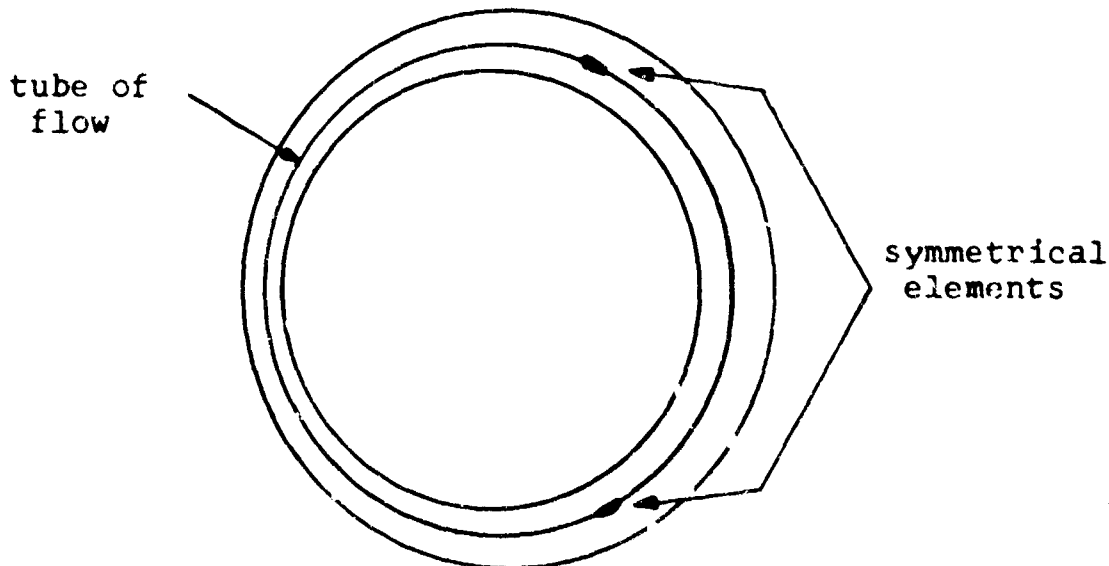
$$p = -\frac{\rho V_\infty^2}{2} - \rho \Phi + \mu \int_0^s \mathbf{l}_s \cdot \nabla^2 \mathbf{V}_R ds \quad (201)$$

$$- \rho \nu (\beta - \nu) [r_2 - \lambda(r_2 - a)]^2 + \rho \nu (\beta - \nu) a^2 + c_0.$$

NOTE. In the term $\mu \int_0^s \mathbf{l}_s \cdot \nabla^2 \mathbf{V}_R ds$ in Eq. 181 the integration extends from the fixed point for which $\theta=0$ on a line of flow around clockwise to point s . The 180-deg rotation of the flow pattern hence adds half the line of flow to the path of integration of the above integral in determining its value for the point which corresponds to s after the rotation. The integral will therefore be the same for both s and this corresponding point if and only if it is not altered

by this 180-deg increase in its path of integration. We shall now show that this condition is satisfied.

Noting that $\mu \nabla^2 V_R$ is the body force per unit volume due to viscous friction and that this is reversed in sign if V_R is reversed in direction, let us consider two elementary lengths of the tube of flow $d\lambda dz$, these being of length ds and located symmetrically about the line $\theta=0$. Since the flow in one of these is similar to that which would pertain in the other if its direction were reversed we see that $\mu l_s \cdot \nabla^2 V_R ds$ is the same for both of these elements (see figure below). It follows that the value of $\mu \int l_s \cdot \nabla^2 V_R ds$ taken over



the first quadrant $0 \leq \theta \leq \pi/2$ is the same as its value taken over the fourth quadrant $3\pi/2 \leq \theta \leq 2\pi$ and that its value taken over the second quadrant is the same as its value taken over the third quadrant.

Turning to Eq. 181 we now see that if we move once around the tube of flow p , V_s , and ϕ return to their initial values, while ψ is constant. It follows that the integral must return to its initial value; hence the value of $\mu \int l_s \cdot \nabla^2 V_R ds$ taken over all four quadrants is zero. The above results may be written

$$\begin{aligned} \int_1 &= \int_4, \quad \int_2 = \int_3, \\ \int_1 + \int_2 + \int_3 + \int_4 &= 0 \end{aligned}$$

where the numbers indicate the quadrants over which the integrations are taken. Replacing \int_1 by \int_4 and \int_2 by \int_3 in the last equation we obtain finally

$$\int_3 + \int_4 = 0.$$

Applying this result to the term $\mu \int_0^s l_s \cdot \nabla^2 V_R ds$ in Eq. 181 we see that since the initial point of the path of integration is taken as that for which $\theta=0$, it follows that in computing the value of this integral for the point which corresponds to s after the rotation we may omit the first 180 deg of the path of integration. The integral over the remainder of the path duplicates that for point s , as desired.

SURFACE ANALOGUE OF THE PRINCIPLE OF LEAST CONSTRAINT. DETERMINATION OF b AND c_0

The constraining body force intensity \mathfrak{F} was introduced artificially to force the mercury to flow in the pattern specified by Eq. 85 (p. 56); then the pressure was determined so as to minimize \mathfrak{F} in a least-square sense as indicated by Eq. 177, which expresses the principle of least constraint (pp. 57-64). In a similar manner we shall now introduce a "constraining pressure" \mathfrak{F}_s which acts on the free surface of the mercury from without, and together with the atmospheric pressure p_a and the pressure p_t due to surface tension balances the pressure in the mercury and also the normal compressive stress due to viscosity in the mercury at the free surface, as indicated by the relation

$$\mathfrak{F}_s = p - 2\mu \frac{\partial v_n}{\partial n} - p_t - p_a \quad (202)$$

where n indicates the direction of the outward (into the mercury) normal to the free surface. The remaining parameters, namely c_0 and b will now be determined so as to minimize the constraining pressure \mathfrak{F}_s in a least-square sense over the free mercury surface a_f , as indicated by the relation

$$\text{Surface constraint} = \int_{a_f} \mathfrak{F}_s^2 da = \text{minimum} \quad (203)$$

where da is the element of area of the free mercury surface. This "principle of least surface constraint" is the surface analogue of the principle of least constraint (Eq. 177), which was used in the previous section.

Before proceeding with the determination of c_0 and b it may be well to consider in retrospect the different variational processes that we have used. These may be outlined as follows:

1. For arbitrarily chosen values of ξ and b , the shape of the relative velocity field is specified by Eq. 85, after which the determination of this field is completed using the principle of least viscous frictional power loss.
2. The pressure field is next determined except for the additive constant c_0 using the principle of least constraint.
3. Holding ξ fixed we finally determine c_0 and b by the principle of least surface constraint.

In the previous section the value of b used in determining the pressure field was the actual value determined by 3 above, although at the time we did not know b explicitly in terms of ξ .

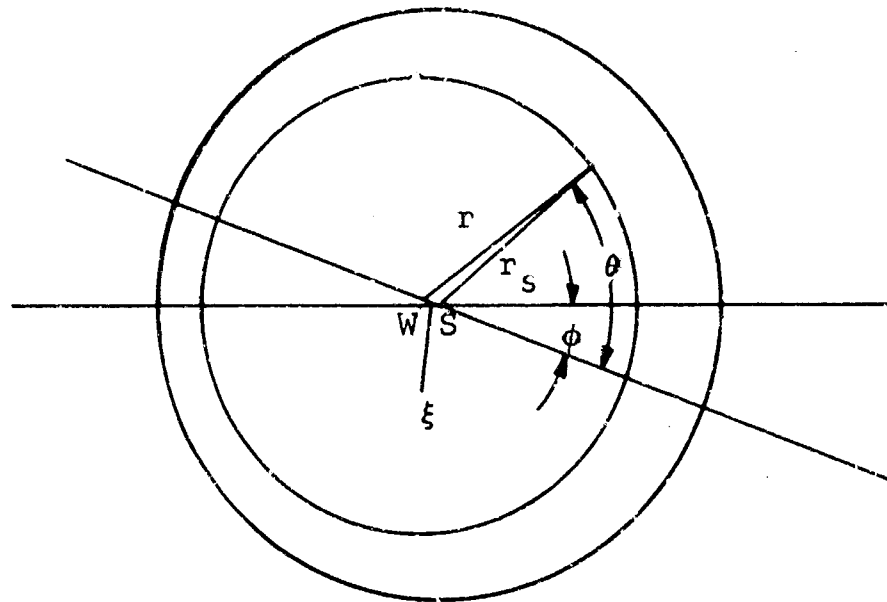
APPLICATION OF THE PRINCIPLE OF LEAST SURFACE CONSTRAINT TO THE DETERMINATION OF b AND c_0

In order to carry out the procedure described in the previous section we must obtain expressions for the various terms which compose \mathcal{F}_s and hence p , which are given by Eqs. 202 and 201, respectively. We shall now consider these terms in order as follows:

The Term $\frac{\rho V_s^2}{2}$. This occurs in p , and follows from Eqs. 110, 156, and 169, thus

$$\frac{\rho V_s^2}{2} = \frac{\rho [q_0(\lambda, z)]^2}{2(r_2 - a)^2} \left\{ 1 + \frac{4\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}, \quad (204)$$

which expression is good through first-order terms in b .



The Term $\rho\Phi$. This appears in p , and is given by Eq. 7, thus, noting the figure above,

$$\rho\Phi = -\frac{\rho}{2} r^2 \beta^2 = -\frac{\rho\beta^2}{2} [r_s^2 + \xi^2 + 2r_s\xi \cos(\theta-\phi)]. \quad (205)$$

Here r_s is given by Eq. 85 with $\lambda=1$, thus

$$r_s = a - b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta. \quad (206)$$

This in Eq. 205 gives, accurate through first-order terms in b ,

$$\begin{aligned} \rho\Phi &= -\frac{\rho\beta^2}{2} \left\{ a^2 + 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta + \xi^2 \right. \\ &\quad \left. + 2\xi \left\{ a - b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta \right\} \cos(\theta-\phi) \right\}, \\ \rho\Phi &= -\frac{\rho\beta^2}{2} \left\{ a^2 + \xi^2 + 2\xi a \cos(\theta-\phi) \right. \\ &\quad \left. + b \cos\theta \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[-2a - 2\xi \cos(\theta-\phi) \right] \right\}. \quad (207) \end{aligned}$$

Omitting the higher order term which contains ξb this becomes

$$\rho\Phi = -\frac{\rho\beta^2 a^2}{2} + \rho a\beta^2 \left\{ b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos\theta - \xi \cos(\theta - \phi) \right\} - \frac{\rho\beta^2 \xi^2}{2}. \quad (208)$$

Here it is likely that the last term in Eq. 208 is of higher order and can be omitted; but this can be decided later.

The Term $\mu \int_0^S l_s \cdot \nabla^2 V_R ds$. This occurs in p, and can be conveniently determined using the notation and choice of axes described in the section on pages 38-48 and shown in the figures on pages 38 and 39. Proceeding thus we obtain

$$\begin{aligned} V_R &= iV_x + jV_y + kV_z, \quad l_s = i \text{ at the origin,} \\ \nabla^2 V_R &= i\nabla^2 V_x + j\nabla^2 V_y + k\nabla^2 V_z, \\ l_s \cdot \nabla^2 V_R &= \nabla^2 V_x = \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}; \end{aligned} \quad (209)$$

and must now determine the three terms which compose this Laplacean. The values of these three terms are, however, required only at the origin.

Differentiating Eq. 114 now gives

$$\begin{aligned} \frac{\partial V_x}{\partial x} &= \frac{\partial V_s}{\partial x} \cos\alpha - V_s \sin\alpha \frac{\partial \alpha}{\partial x}, \\ \frac{\partial^2 V_x}{\partial x^2} &= \frac{\partial^2 V_s}{\partial x^2} \cos\alpha - \frac{\partial V_s}{\partial x} \sin\alpha \frac{\partial \alpha}{\partial x} - \frac{\partial V_s}{\partial x} \sin\alpha \frac{\partial \alpha}{\partial x} \\ &\quad - V_s \cos\alpha \left(\frac{\partial \alpha}{\partial x} \right)^2 - V_s \sin\alpha \frac{\partial^2 \alpha}{\partial x^2}. \end{aligned} \quad (210)$$

At the origin, which is the only point in which we are interested, $\alpha=0$, and Eq. 210 becomes

$$\frac{\partial^2 V_x}{\partial x^2} = \frac{\partial^2 V_s}{\partial x^2} - V_s \left(\frac{\partial \alpha}{\partial x} \right)^2. \quad (211)$$

Similarly, at the origin

$$\frac{\partial^2 V_x}{\partial y^2} = \frac{\partial^2 V_s}{\partial y^2} - V_s \left(\frac{\partial \alpha}{\partial y} \right)^2, \quad (212)$$

$$\frac{\partial^2 V_x}{\partial z^2} = \frac{\partial^2 V_s}{\partial z^2} - V_s \left(\frac{\partial \alpha}{\partial z} \right)^2. \quad (213)$$

In Eqs. 211, 212, and 213 V_s is given by Eq. 110; and the following directional derivatives

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial s}, \quad \frac{\partial \alpha}{\partial y} = \frac{\partial \alpha}{\partial s_n}, \quad \frac{\partial \alpha}{\partial z}, \quad (214)$$

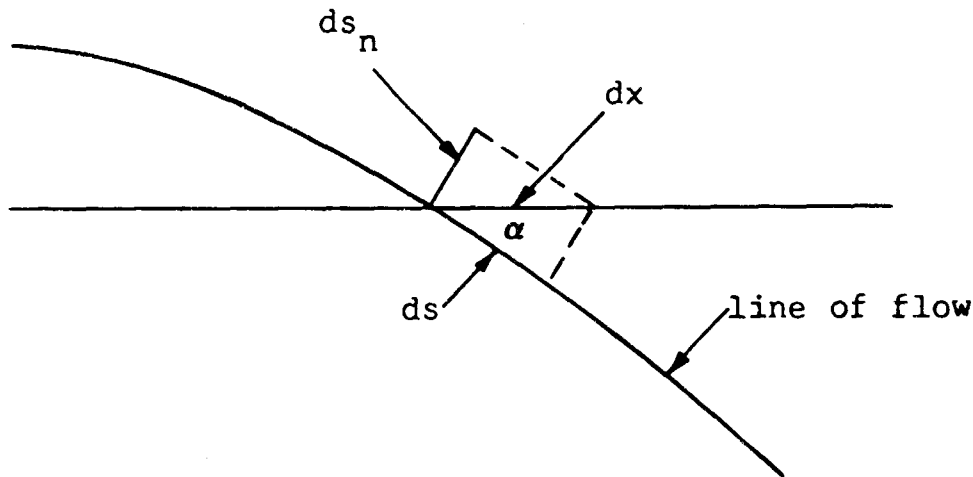
which are evaluated at the origin, are available from Eqs. 121, 134, and 148, respectively. Also $\frac{\partial^2 V_s}{\partial z^2}$ is available from Eq. 110. Our problem hence consists mainly of determining $\frac{\partial^2 V_s}{\partial x^2}$ and $\frac{\partial^2 V_s}{\partial y^2}$.

At the origin

$$\frac{\partial^2 V_s}{\partial x^2} = \frac{\partial}{\partial s} \left(\frac{\partial V_s}{\partial x} \right) \quad (215)$$

since both sides of this equation express the directional derivative of $\frac{\partial V_s}{\partial x}$ in the direction of a line of flow. Noting that the direction of increasing s_n is outward and normal to the line of flow, we now have

$$\frac{\partial V_s}{\partial x} = \frac{\partial V_s}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial V_s}{\partial s_n} \frac{\partial s_n}{\partial x}, \quad (216)$$



which, noting figure above, may be written

$$\frac{\partial V_s}{\partial x} = \frac{\partial V_s}{\partial s} \cos \alpha + \frac{\partial V_s}{\partial s_n} \sin \alpha. \quad (217)$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{\partial V_s}{\partial x} \right) &= \frac{\partial^2 V_s}{\partial s^2} \cos \alpha - \frac{\partial V_s}{\partial s} \sin \alpha \frac{\partial \alpha}{\partial s} \\ &+ \frac{\partial^2 V_s}{\partial s \partial s_n} \sin \alpha + \frac{\partial V_s}{\partial s_n} \cos \alpha \frac{\partial \alpha}{\partial s}. \end{aligned}$$

At the origin, where $\alpha=0$, this becomes, noting Eq. 215,

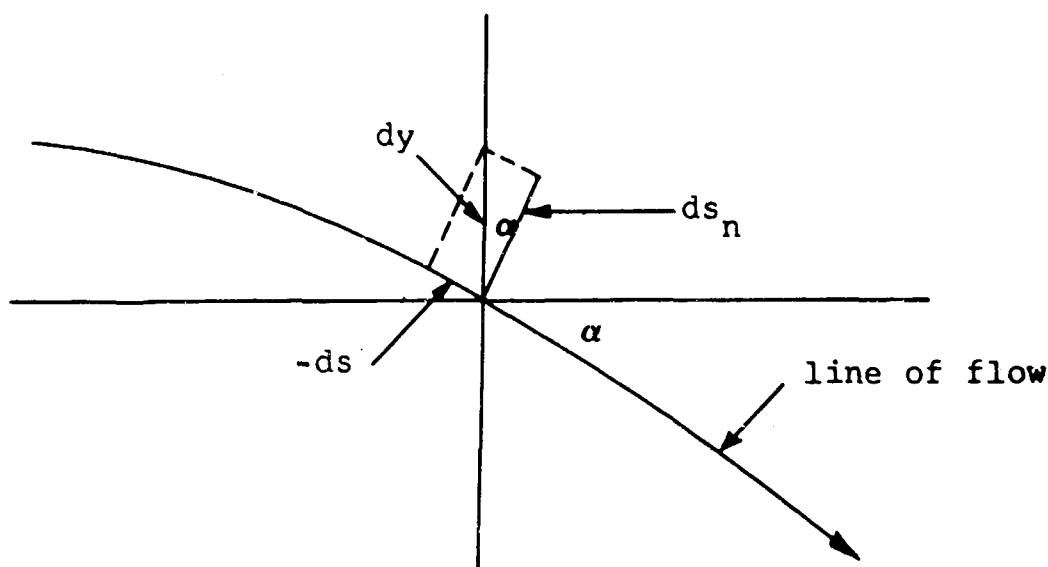
$$\frac{\partial^2 V_s}{\partial x^2} = \frac{\partial^2 V_s}{\partial s^2} + \frac{\partial V_s}{\partial s_n} \frac{\partial \alpha}{\partial s}. \quad (218)$$

In a similar manner we have at the origin

$$\frac{\partial^2 V_s}{\partial y^2} = \frac{\partial}{\partial s_n} \left(\frac{\partial V_s}{\partial y} \right), \quad (219)$$

since both sides of this equation express the directional derivative of $\frac{\partial V_s}{\partial y}$ in the direction of the outward normal to the line of flow.

$$\frac{\partial V_s}{\partial y} = \frac{\partial V_s}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial V_s}{\partial s_n} \frac{\partial s_n}{\partial y}, \quad (220)$$



or, noting the figure above,

$$\frac{\partial V_s}{\partial y} = - \frac{\partial V_s}{\partial s} \sin \alpha + \frac{\partial V_s}{\partial s_n} \cos \alpha, \quad (221)$$

$$\begin{aligned} \frac{\partial}{\partial s_n} \left(\frac{\partial V_s}{\partial y} \right) &= - \frac{\partial^2 V_s}{\partial s_n \partial s} \sin \alpha - \frac{\partial V_s}{\partial s} \cos \alpha \frac{\partial \alpha}{\partial s_n} \\ &\quad + \frac{\partial^2 V_s}{\partial s_n^2} \cos \alpha - \frac{\partial V_s}{\partial s_n} \sin \alpha \frac{\partial \alpha}{\partial s_n}. \end{aligned}$$

At the origin, where $\alpha=0$, this becomes, noting Eq. 219,

$$\frac{\partial^2 V_s}{\partial y^2} = \frac{\partial^2 V_s}{\partial s_n^2} - \frac{\partial V_s}{\partial s} \frac{\partial \alpha}{\partial s_n}. \quad (222)$$

Equations 218 and 222 in Eqs. 211 and 212, respectively, now give

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V_s}{\partial s^2} + \frac{\partial V_s}{\partial s_n} \frac{\partial \alpha}{\partial s} - V_s \left(\frac{\partial \alpha}{\partial x} \right)^2, \quad (223)$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V_s}{\partial s_n^2} - \frac{\partial V_s}{\partial s} \frac{\partial \alpha}{\partial s_n} - V_s \left(\frac{\partial \alpha}{\partial y} \right)^2. \quad (224)$$

Continuing with Eq. 223 we note that $\frac{\partial V_s}{\partial s}$ is given as a function of λ , z , and θ by Eq. 139. Along a line of flow λ and z are constant; hence

$$\frac{\partial^2 V_s}{\partial s^2} = \left[\frac{\partial}{\partial \theta} \left(\frac{\partial V_s}{\partial s} \right) \right] \frac{\partial \theta}{\partial s}.$$

Applying Eqs. 138 and 139 this becomes, good through first-order terms in b ,

$$\frac{\partial^2 V_s}{\partial s^2} \cong \frac{2q(\lambda, z)\lambda b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta}{(r_2 - a)^2 [r_2 - \lambda(r_2 - a)]^2}, \quad (225)$$

or, replacing $q(\lambda, z)$ by Eq. 156 and hence, in effect by Eq. 161,

$$\frac{\partial^2 V_s}{\partial s^2} \cong \frac{2\lambda b(\beta - \nu) \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta}{(r_2 - a) [r_2 - \lambda(r_2 - a)]}. \quad (226)$$

The second term of Eq. 223 is given by Eq. 142 in which $q(\lambda, z)$ is replaced by Eq. 156 and hence Eq. 161, and by Eq. 121, which is $\frac{1}{R}$, and Eq. 100; thus

$$\begin{aligned} \frac{\partial V_s}{\partial s_n} \frac{\partial \alpha}{\partial s} \cong & \frac{(\beta-\nu)}{r_2 - \lambda(r_2 - a)} \left\{ 1 - \frac{4\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \\ & + \frac{2(\beta-\nu)b}{(r_2 - a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta. \end{aligned} \quad (227)$$

The third term of Eq. 223 follows from Eq. 110, in which $q(\lambda, z)$ is replaced by Eq. 156 and hence Eq. 161, and from Eq. 121, which is $\frac{1}{R}$, and Eq. 100; thus

$$V_s \left(\frac{\partial \alpha}{\partial x} \right)^2 \cong \frac{(\beta-\nu)}{r_2 - \lambda(r_2 - a)} \left\{ 1 - \frac{2\lambda b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}. \quad (228)$$

Substituting Eqs. 226, 227, and 228 in Eq. 223, we obtain finally

$$\frac{\partial^2 V_x}{\partial x^2} \cong \frac{2(\beta-\nu)b}{(r_2 - a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \quad (229)$$

good through first-order terms in b .

Next, continuing with Eq. 224 we note that $\frac{\partial V_s}{\partial s_n}$ is given by Eq. 142; also $\frac{\partial^2 V_s}{\partial s_n^2}$ can be obtained by replacing V_s by $\frac{\partial V_s}{\partial s_n}$ in Eqs. 140 through 141. Since, due to Eq. 85,

$$\sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta} \right)^2} \cong 1,$$

we thus obtain

$$\frac{\partial^2 V_s}{\partial s_n^2} \cong \frac{\frac{\partial}{\partial \lambda} \left(\frac{\partial V_s}{\partial s_n} \right)}{\frac{\partial r}{\partial \lambda}} - \frac{1}{r^2} \left[\frac{\partial}{\partial \theta} \left(\frac{\partial V_s}{\partial s_n} \right) \right] \frac{\partial r}{\partial \theta}. \quad (230)$$

Substituting Eqs. 85 and 142 in Eq. 230 we see that the second term in Eq. 230 is of second order in b , and can hence be omitted. In the first term $q(\lambda, z)$ is replaced by Eq. 156 and hence Eq. 161, and we obtain finally

$$\frac{\partial^2 V_s}{\partial s_n^2} \cong \frac{6(\beta-\nu)b}{(r_2-a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta. \quad (231)$$

Noting Eq. 214 we substitute Eq. 231 and the expression for $\frac{\partial \alpha}{\partial s_n}$ given by Eq. 134 in Eq. 224, and thus obtain

$$\frac{\partial^2 V_x}{\partial y^2} = - \frac{2}{V_s} \left(\frac{\partial V_s}{\partial s} \right)^2 + \frac{6(\beta-\nu)b}{(r_2-a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta. \quad (232)$$

From Eqs. 110 and 139, however, we see that the first term on the right-hand side of Eq. 232 is of second order in b ; hence good through first-order terms in b we have

$$\frac{\partial^2 V_x}{\partial y^2} \cong \frac{6(\beta-\nu)b}{(r_2-a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta. \quad (233)$$

Turning finally to Eq. 213 we replace $q(\lambda, z)$ in Eq. 110 by Eq. 156, and hence Eq. 161. Differentiating twice with respect to z and placing $\lambda=1$, corresponding to the inner free surface, we then obtain for the first term in Eq. 213

$$\frac{\partial^2 V_s}{\partial z^2} \cong \frac{4ab(\beta-\nu)}{(r_2-a)z_1^2} \cos \theta. \quad (234)$$

Next we see from Eqs. 110 and 148 that the second term in Eq. 213 is of second order in b , and can hence be omitted. Good through first-order terms in b we thus have

$$\frac{\partial^2 V_x}{\partial z^2} \cong \frac{4ab(\beta-\nu)}{(r_2-a)z_1^2} \cos \theta. \quad (235)$$

Putting $\lambda=1$ in Eq. 85 we obtain

$$ds = - \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} d\theta \cong - r d\theta$$

$$ds \cong - \left\{ a - b \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \cos \theta \right\} d\theta \quad (236)$$

for the element of a flow line in the free surface. Substituting Eqs. 229, 233, and 235 in the last equation of Eq. 209; multiplying by Eq. 236 and integrating, we obtain finally

$$\mu \int_0^s l_s \cdot \nabla^2 V_R ds \cong - \frac{4\mu a^2 b(\beta-\nu)}{(r_2-a)z_1^2} \sin \theta$$

$$- \frac{8\mu ab(\beta-\nu)}{(r_2-a)^2} \left[1 - \left(\frac{z}{z_1}\right)^2 \right] \sin \theta \quad (237)$$

good through first-order terms in b .

The Term $2\mu \frac{\partial V}{\partial n}$. This occurs in δ_s . At any point of the free mercury surface let l_n be a fixed unit normal vector pointing outward (into the mercury). Then

$$V_n = l_n \cdot V_R = V_s l_s \cdot l_n \quad (238)$$

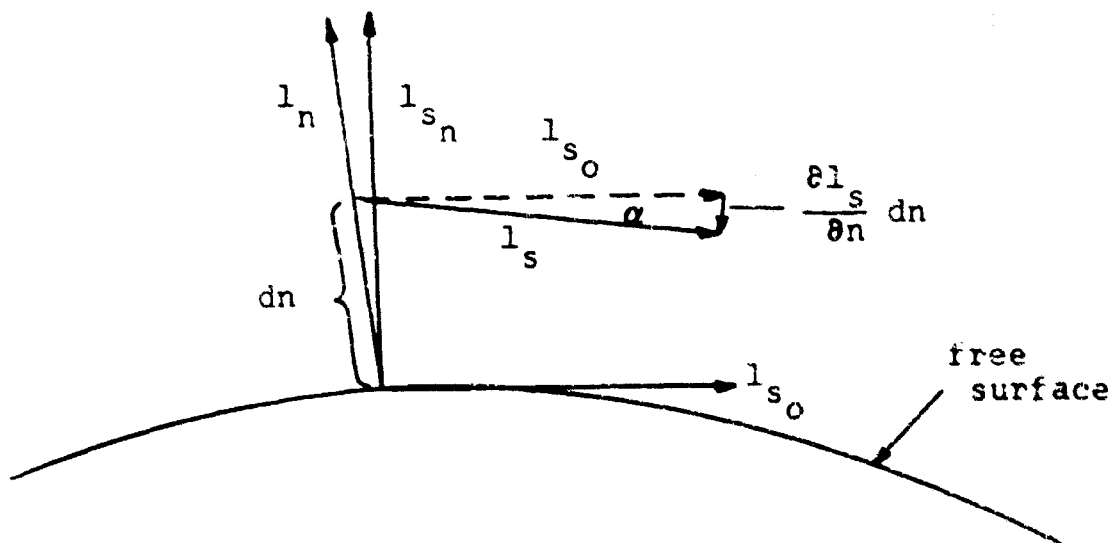
since $V_R = V_s l_s$. It follows that

$$\frac{\partial V}{\partial n} = \frac{\partial V}{\partial n} l_s \cdot l_n + V_s \frac{\partial}{\partial n} (l_n \cdot l_s). \quad (239)$$

It is necessary to determine this quantity only on the free surface. On this surface $l_n \perp l_s$; hence the first term vanishes. In the second term we note that along the normal l_n remains constant, whereas in general l_s does not. Eq. 239 thus becomes

$$\frac{\partial V}{\partial n} = V_s l_n \cdot \frac{\partial l_s}{\partial n}. \quad (240)$$

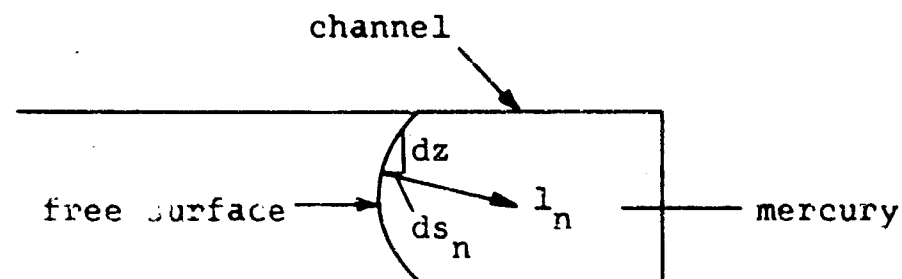
In order to obtain this dot product we shall determine components along the directions of the mutually orthogonal vectors l_{s_0} , l_{s_n} , l_z , these vectors being evaluated on the free surface at the base of the normal. Here the subscript o has been added to s to avoid confusing the fixed vector l_{s_0} with the variable vector l_s , which at the free surface becomes l_{s_0} , as shown below.



Although l_s is variable, its component along l_z is everywhere zero; furthermore it is evident that $dl_s \cdot l_{s_0} = 0$. It follows that the only nonvanishing component of $\frac{\partial l_s}{\partial n}$ is that along l_{s_n} . In Eq. 240 we therefore need only that component of l_n which lies along l_{s_n} . The component along l_{s_0} is, of course, zero, since l_{s_0} is tangent to the free surface.

Noting the following figure and the fact that $l_n \cdot l_{s_0} = 0$ we see that

$$\text{Component of } l_n \text{ along } l_{s_n} = \cos \angle l_n l_{s_n} = \left| \frac{dz}{\sqrt{ds_n^2 + dz^2}} \right|. \quad (241)$$



Moving along ds_n from the free surface to the lower end of dz in the figure above, r and θ change by amounts dr and $d\theta$ given in terms of $d\lambda$ by Eq. 106. Starting at the same point and moving along the free surface to the upper end of dz we have $\lambda=1$, $d\lambda=0$, and

$$dr = \frac{\partial r}{\partial z} dz + \frac{\partial r}{\partial \theta} d\theta, \quad (242)$$

r being given by Eq. 85. Since dz is vertical dr and $d\theta$ are the same as before; hence Eq. 242 serves to determine dz . Substituting Eq. 106 in Eq. 242, we obtain

$$\frac{r^2 \frac{\partial r}{\partial \lambda}}{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} d\lambda = \frac{\partial r}{\partial z} dz + \frac{\partial r}{\partial \theta} \left[- \frac{\frac{\partial r}{\partial \theta} \frac{\partial r}{\partial \lambda}}{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} d\lambda \right]$$

$$\frac{\partial r}{\partial \lambda} d\lambda = \frac{\partial r}{\partial z} dz$$

$$dz = \frac{\frac{\partial r}{\partial \lambda}}{\frac{\partial r}{\partial z}} d\lambda \quad (243)$$

where $d\lambda$ is the change in λ corresponding to ds_n . Equations 107 and 243 in Eq. 241 now give

$$\text{Component of } l_n \text{ along } l_{s_n} = \frac{1}{\left| \frac{\partial r}{\partial z} \right| \sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta} \right)^2 + \left(\frac{\partial r}{\partial z} \right)^2}}$$

$$\text{Component of } l_n \text{ along } l_{s_n} = \frac{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}{\sqrt{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2}} \quad (244)$$

Noting Eq. 85 it now follows that

$$\text{Component of } l_n \text{ along } l_{s_n} \cong 1 \quad (245)$$

good through first-order terms in b .

Next we shall determine the component of l_s along l_{s_n} . Noting the figure on page 75 we see that

$$\text{Component of } l_s \text{ along } l_{s_n} = -\sin\alpha; \quad (246)$$

however, since $\alpha=0$ at the free surface we have in effect

$$\alpha = d\alpha = \frac{\partial \alpha}{\partial s_n} (ds_n)_{\text{normal}} + \frac{\partial \alpha}{\partial z} (dz)_{\text{normal}} \quad (247)$$

where the word "normal" has been added to stress the fact that these quantities correspond to an element dn of the normal line, and not to an element of length in the free surface, as shown in the figure on page 76. However, if the triangle of legs ds_n and dz in the figure on page 76 be rotated 90 deg clockwise, we see that the quantities dz and $(-ds_n)$ in the figure on page 76 can play the role of $(ds_n)_{\text{normal}}$ and $(dz)_{\text{normal}}$, respectively, in Eq. 247, in which case, noting Eqs. 107 and 243,

$$dn = \sqrt{ds_n^2 + dz^2} = \frac{\partial r}{\partial \lambda} \sqrt{\frac{1}{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2} + \frac{1}{\left(\frac{\partial r}{\partial z}\right)^2}} d\lambda$$

$$dn = \frac{\frac{\partial r}{\partial \lambda}}{\left|\frac{\partial r}{\partial z}\right|} \sqrt{\frac{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2}{1 + \left(\frac{1}{r} \frac{\partial r}{\partial \theta}\right)^2}} d\lambda \cong \frac{\frac{\partial r}{\partial \lambda}}{\left|\frac{\partial r}{\partial z}\right|} d\lambda, \quad (248)$$

or, noting Eq. 243,

$$dn \approx |dz|. \quad (249)$$

Equation 247 now gives

$$\frac{d\alpha}{dn} = \frac{\partial \alpha}{\partial s_n} \frac{|dz|}{dn} \mp \frac{\partial \alpha}{\partial z} \frac{ds_n}{dn}, \quad (250)$$

wherein the + sign would be obtained if l_n sloped upward in the figure on page 76, in which case the triangle would be rotated 90 deg counterclockwise. Noting Eqs. 107, 248, and 249 this becomes

$$\frac{d\alpha}{dn} \approx \frac{\partial \alpha}{\partial s_n} - \frac{\partial \alpha}{\partial z} \frac{\partial r}{\partial z}. \quad (251)$$

From Eqs. 85 and 149 we now see that the second term is of second order in b , and can hence be omitted. The first term is given by Eq. 134, thus

$$\frac{d\alpha}{dn} \approx \frac{1}{V_s} \frac{\partial V_s}{\partial s}. \quad (252)$$

Since from Eqs. 246 and 247

$$\text{Component of } \frac{dl_s}{dn} \text{ along } l_{s_n} = \frac{\sin \alpha}{dn} = - \frac{\sin d\alpha}{dn} = - \frac{d\alpha}{dn} \quad (253)$$

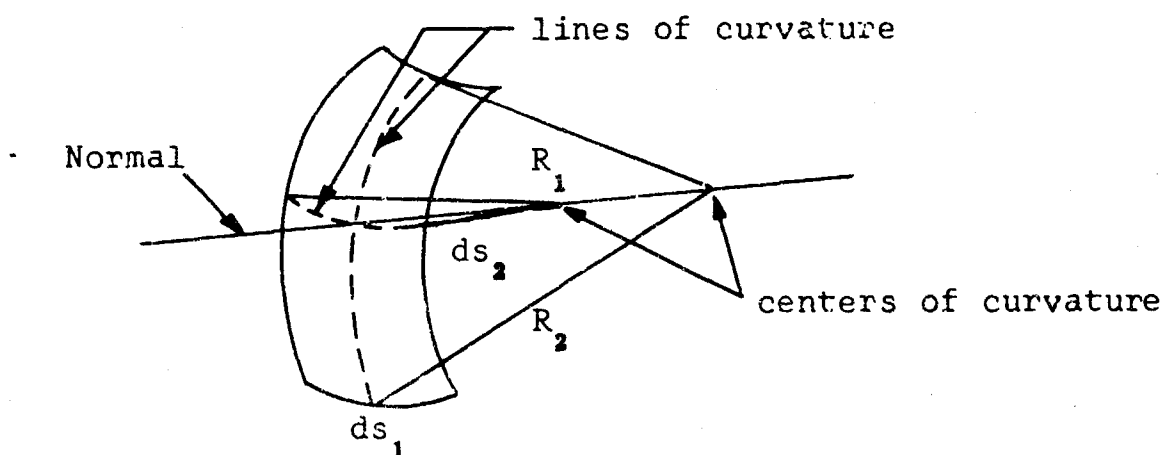
at the free surface, it now follows from Eqs. 240, 245, 246, 252, and 253 that

$$\frac{\partial V_n}{\partial n} = - \frac{\partial V_s}{\partial s}. \quad (254)$$

Substituting Eq. 139 in which $q(\lambda, z)$ has been replaced by Eq. 156 and hence Eq. 161, we obtain finally

$$2\mu \frac{\partial V_n}{\partial n} \approx \frac{4\mu b(\beta - \nu)}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta. \quad (255)$$

The Term p_t .⁵ This occurs in \mathfrak{F}_5 . Using the lines of curvature as parametric lines let us consider an element of area of the free mercury surface, shown in the figure below.



Denoting the surface tension by T , this being the force per unit length transmitted across any line in the surface, the normal component of the force exerted by surface tension along the sides of length ds_1 is

$$\begin{array}{l} \text{Normal component} \\ \text{of forces on sides of length } ds_1 \end{array} = T ds_1 \frac{ds_2}{R_2} \quad (256)$$

where R_2 is a principal radius of curvature. Here the reasoning is essentially the same as in the case of a vibrating string or membrane. Adding to Eq. 256 a similar expression for the normal component of the forces transmitted along the sides of length ds_2 , then dividing by the area $ds_1 ds_2$ we obtain for the pressure due to surface tension

⁵ The effect of surface tension is considered here in connection with the free surface given by Eq. 85 with $\lambda=1$; however, surface tension was not used directly in determining this surface.

$$p_t = T \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = 2T\mathfrak{M} \quad (257)$$

where \mathfrak{M} is the mean curvature of the surface.

In order to compute the mean curvature of the free mercury surface we place $\lambda=1$ in Eq. 85, use z and θ as parameters, and have as the equations of the free surface

$$\begin{aligned} x &= r \cos \theta = a \cos \theta - b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos^2 \theta, \\ y &= r \sin \theta = a \sin \theta - b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \sin \theta, \\ z &= z. \end{aligned} \quad (258)$$

Differentiating we obtain

$$\begin{aligned} \frac{\partial x}{\partial z} &= \frac{2bz}{z_1^2} \cos^2 \theta, \\ \frac{\partial y}{\partial z} &= \frac{2bz}{z_1^2} \cos \theta \sin \theta, \\ \frac{\partial z}{\partial z} &= 1; \\ \frac{\partial x}{\partial \theta} &= -a \sin \theta + 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \sin \theta, \\ \frac{\partial y}{\partial \theta} &= a \cos \theta + b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\sin^2 \theta - \cos^2 \theta), \\ \frac{\partial z}{\partial \theta} &= 0. \end{aligned} \quad (259)$$

Accurate through first-order terms in b the quantities E , F , and G in the first fundamental form are hence⁶

⁶Struik, Dirk Jan. Lectures on Classical Differential Geometry. Cambridge, Mass., Addison-Wesley Press, 1950, pp. 58, 75, and 83.

$$E \cong 1,$$

$$F \cong \frac{2abz}{z_1^2} (-\cos^2 \theta \sin \theta + \cos^2 \theta \sin \theta) = 0, \quad (260)$$

$$G \cong a^2 + 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (-2 \sin^2 \theta \cos \theta + \sin^2 \theta \cos \theta - \cos^3 \theta)$$

$$G \cong a^2 - 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta;$$

also

$$EG - F^2 \cong G \cong a^2 - 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta, \quad (261)$$

from which we obtain as the element of surface area

$$da_f = \sqrt{EG - F^2} dz d\theta \cong \left\{ a - b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} dz d\theta. \quad (262)$$

Taking second derivatives we now obtain

$$\frac{\partial^2 x}{\partial z^2} = \frac{2b}{z_1^2} \cos^2 \theta,$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{2b}{z_1^2} \cos \theta \sin \theta,$$

$$\frac{\partial^2 z}{\partial z^2} = 0;$$

$$\frac{\partial^2 x}{\partial z \partial \theta} = - \frac{4bz}{z_1^2} \cos \theta \sin \theta,$$

$$\frac{\partial^2 y}{\partial z \partial \theta} = \frac{2bz}{z_1^2} (\cos^2 \theta - \sin^2 \theta),$$

(263)

$$\frac{\partial^2 z}{\partial z \partial \theta} = 0;$$

$$\frac{\partial^2 x}{\partial \theta^2} = -a \cos \theta + 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\cos^2 \theta - \sin^2 \theta),$$

$$\frac{\partial^2 y}{\partial \theta^2} = -a \sin \theta + 4b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta \cos \theta,$$

$$\frac{\partial^2 z}{\partial \theta^2} = 0.$$

Using Eqs. 259 and 263 we now set up the third-order determinants which give e, f, and g of the second fundamental form. In each case the last column is 0, 1, 0; hence developing by minors with respect to this column, and omitting terms which contribute only terms of order two or higher in b to the expansion of the determinant, we obtain

$$e \cong - \begin{vmatrix} \frac{2b}{z_1^2} \cos^2 \theta & \frac{2b}{z_1^2} \cos \theta \sin \theta \\ -a \sin \theta & a \cos \theta \end{vmatrix} + \sqrt{EG-F^2}$$

$$e \cong - \frac{2ab}{z_1^2} \cos \theta + \sqrt{EG-F^2}, \quad (264)$$

$$f \cong - \begin{vmatrix} -\frac{4bz}{z_1^2} \cos \theta \sin \theta & \frac{2bz}{z_1^2} (\cos^2 \theta - \sin^2 \theta) \\ -a \sin \theta & a \cos \theta \end{vmatrix} + \sqrt{EG-F^2}$$

$$f \cong \frac{2abz}{z_1^2} \sin \theta (2\cos^2 \theta - \cos^2 \theta + \sin^2 \theta) + \sqrt{EG-F^2}$$

$$f \cong \frac{2abz}{z_1^2} \sin \theta + \sqrt{EG-F^2}, \quad (265)$$

$$g = - \begin{vmatrix} -a \cos \theta + 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\cos^2 \theta - \sin^2 \theta) & -a \sin \theta + 4b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta \\ -a \sin \theta + 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \sin \theta & a \cos \theta + b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\sin^2 \theta) \end{vmatrix} + \sqrt{EG-F^2}$$

$$g \approx \left\{ a^2 - 6ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin^2 \theta \cos \theta - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\cos^2 \theta - \sin^2 \theta) \cos \theta \right\} + \sqrt{EG}$$

$$g \approx \left\{ a^2 - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta (2 \sin^2 \theta + \cos^2 \theta - \sin^2 \theta) \right\} + \sqrt{EG-F^2}$$

$$g \approx \left\{ a^2 - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} + \sqrt{EG-F^2}.$$

The mean curvature is now given by

$$- \kappa = \frac{gE - 2fF + eG}{2(EG-F^2)},$$

$$- \kappa \approx \frac{a^2 - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta - \frac{2a^3 b}{z_1^2} \cos \theta}{2 \left\{ a^2 - 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}^{3/2}}$$

$$- \kappa \approx \frac{1}{2} \left\{ a^2 - ab \left[3 - 3 \left(\frac{z}{z_1} \right)^2 + 2 \left(\frac{a}{z_1} \right)^2 \right] \cos \theta \right\} \left\{ \frac{1}{a^3} + \frac{3b}{a^4} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}$$

$$- \kappa \approx \frac{1}{2a} \left(1 - \frac{2ab}{z_1^2} \cos \theta \right).$$

Substituting in Eq. 257 we now obtain finally

$$p_t \approx - \frac{T}{a} \left(1 - \frac{2ab}{z_1^2} \cos \theta \right).$$

A

NA

ss

n²

EG

cc

$$\begin{array}{l} 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\cos^2 \theta - \sin^2 \theta) - a \sin \theta + 4b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta \cos \theta \\ 2b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \sin \theta a \cos \theta + b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\sin^2 \theta - \cos^2 \theta) \end{array}$$

$$\left(\frac{z}{z_1} \right)^2 \left[\sin^2 \theta \cos \theta - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] (\cos^2 \theta - \sin^2 \theta) \cos \theta \right] + \sqrt{EG - F^2}$$

$$\left(\frac{z}{z_1} \right)^2 \left[\cos \theta (2 \sin^2 \theta + \cos^2 \theta - \sin^2 \theta) \right] + \sqrt{EG - F^2}$$

$$\left(\frac{z}{z_1} \right)^2 \left[\cos \theta \right] + \sqrt{EG - F^2} . \quad (266)$$

ure is now given by

$$- \mathfrak{R} = \frac{gE - 2fF + eG}{2(EG - F^2)} , \quad (267)$$

$$\frac{a^3 - 3ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta - \frac{2a^3 b}{z_1^3} \cos \theta}{2 \left\{ a^3 - 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\}^{3/2}}$$

$$\frac{1}{2} \left\{ a^3 - ab \left[3 - 3 \left(\frac{z}{z_1} \right)^2 + 2 \left(\frac{a}{z_1} \right)^2 \right] \cos \theta \right\} \left\{ \frac{1}{a^3} + \frac{3b}{a^4} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \\ \frac{1}{2a} \left(1 - \frac{2ab}{z_1^2} \cos \theta \right) . \quad (268)$$

257 we now obtain finally

$$p_t \approx - \frac{T}{a} \left(1 - \frac{2ab}{z_1^2} \cos \theta \right) . \quad (269)$$

B

Placing Eqs. 204, 208, and 237 in Eq. 201; and then placing Eqs. 201, 255, and 269 in Eq. 202, we now obtain

$$\begin{aligned} \bar{v}_s \cong & - \frac{\rho a^2 (\beta - \nu)^2}{2} \left\{ 1 - \frac{4b}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \\ & + \frac{\rho \beta^2 a^2}{2} - \rho a \beta^2 \left\{ b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right. \\ & \left. - \xi \cos(\theta - \phi) \right\} + \frac{\rho \beta^2 \xi^2}{2} - \frac{4\mu a^2 b (\beta - \nu)}{(r_2 - a) z_1^2} \sin \theta \\ & - \frac{8\mu a b (\beta - \nu)}{(r_2 - a)^2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta \\ & + c_0 - \frac{4\mu b (\beta - \nu)}{r_2 - a} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \sin \theta \\ & + \frac{T}{a} \left(1 - \frac{2ab}{z_1^2} \cos \theta \right) - p_a \end{aligned} \quad (270)$$

accurate through first-order terms in b . This may be written

$$\bar{v}_s = \bar{A} + \bar{B} \sin \theta + \bar{C} \cos \theta \quad (271)$$

where

$$\begin{aligned} \bar{A} = & - \frac{\rho a^2 (\beta - \nu)^2}{2} + \frac{\rho \beta^2 a^2}{2} + \frac{\rho \beta^2 \xi^2}{2} + c_0 + \frac{T}{a} - p_a \\ \bar{A} = & \frac{\rho}{2} (2a^2 \beta \nu - a^2 \nu^2 + \beta^2 \xi^2) + \frac{T}{a} - p_a + c_0, \end{aligned} \quad (272)$$

$$\begin{aligned} \bar{B} = & \rho a \beta^2 \xi \sin \phi - \frac{4\mu b (\beta - \nu)}{(r_2 - a)^2} (r_2 + a) \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2 - a}{r_2 + a} \right) \right. \\ & \left. - \left(\frac{z}{z_1} \right)^2 \right], \end{aligned} \quad (273)$$

$$\begin{aligned} \bar{C} = \rho ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[\frac{2a(\beta-\nu)^2}{r_s^2 - a^2} - \beta^2 \right] \\ + \rho a \beta^2 \xi \cos \phi - \frac{2Tb}{z_1^2} \end{aligned} \quad (274)$$

If, noting Eq. 262, we also place

$$\bar{D} = b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] ; \quad (275)$$

then Eq. 203 becomes

$$\begin{aligned} \kappa_s = \int_{-z_1}^{z_1} \int_0^{2\pi} (\bar{A} + \bar{B} \sin \theta + \bar{C} \cos \theta)^2 (a - \bar{D} \cos \theta) dz d\theta \\ = \text{minimum} \end{aligned} \quad (276)$$

where κ_s is the surface constraint. Multiplying out the first factor and carrying out the integration on θ now gives

$$\kappa_s = \pi \int_{-z_1}^{z_1} [a(2\bar{A}^2 + \bar{B}^2 + \bar{C}^2) - 2\bar{A}\bar{C}\bar{D}] dz = \text{minimum.} \quad (277)$$

Next we note that in Eq. 39 P is given by Eq. 173 accurate through third-order terms in b ; and, noting Eq. 92 and the figure on page 66, $(-y_0)$ is Eq. 93 times $\sin \phi$, accurate through second-order terms in b . Equation 39 thus becomes

$$P_s b^2 = \frac{2Mba^2 \beta^2 (\beta-\nu) \xi}{3(r_s^2 - a^2)} \sin \phi \quad (278)$$

wherein both sides are accurate through second-order terms in b . Later we shall see that due to Eq. 291 they are accurate through third-order terms in b . Substituting from Eqs. 83 and 172 this becomes

$$\beta^2 \xi \sin \phi = K \mu (\beta-\nu) b \quad (279)$$

where

$$K = \frac{1}{15\rho a^2} \left\{ 24 \left[2 \left(\frac{r_2}{r_2-a} \right)^3 - 5 \left(\frac{r_2}{r_2-a} \right)^2 + 8 \left(\frac{r_2}{r_2-a} \right) - 4 \right] + \left(\frac{r_2}{z_1} \right)^2 \left[40 \left(\frac{r_2}{r_2-a} \right) - 90 + 78 \left(\frac{r_2-a}{r_2} \right) - 25 \left(\frac{r_2-a}{r_2} \right)^2 \right] \right\} . \quad (280)$$

From Eq. 279 it follows that

$$\beta^2 \xi \cos \phi = \pm \sqrt{\beta^2 \xi^2 - K^2 \mu^2 (\beta - \nu)^2 b^2} . \quad (281)$$

We regard ξ as given, whereupon ϕ is determined by Eq. 279, which can be used to eliminate ϕ from Eqs. 273 and 274. c_0 and b are now determined so as to minimize the surface constraint κ_s , given by Eq. 277. Let us first determine c_0 . This we can do by minimizing κ_s for an arbitrary choice of b , which amounts to equating $\frac{\delta \kappa_s}{\delta c_0}$ to zero, κ_s being regarded as a function of c_0 and b . Noting Eqs. 272 through 275 we thus obtain

$$\int_{-z_1}^{z_1} (2a\bar{A} - \bar{C} \bar{D}) dz = 0 \quad (282)$$

as the relation which determines c_0 as a function of b . Multiplying Eq. 282 by $2\pi\bar{A}$, which does not contain z , and subtracting from Eq. 277 we then obtain

$$\int_{-z_1}^{z_1} (-2\bar{A}^2 + \bar{B}^2 + \bar{C}^2) dz = \text{minimum} \quad (283)$$

as the relation for determining b .

Next, noting Eqs. 272 through 275 we carry out the integration in Eq. 282, and thereby obtain

$$\begin{aligned}
4az_1 \bar{A} - \frac{16}{15} z_1 \rho ab^3 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] \\
- \frac{4}{3} z_1 \rho ab \beta^2 \xi \cos \phi + \frac{4}{3} z_1 \frac{2Tb^3}{z_1^2} = 0 \\
\bar{A} = \frac{4}{15} \rho b^2 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] + \frac{1}{3} \rho b \beta^2 \xi \cos \phi - \frac{2Tb^2}{3az_1^2} . \quad (284)
\end{aligned}$$

This expression equated to Eq. 272 determines c_0 , which appears in \bar{A} alone. Substituting Eq. 281 in Eq. 284 we now get

$$\begin{aligned}
\bar{A} = b^2 \left\{ \frac{4}{15} \rho \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2}{3} \frac{T}{az_1^2} \right\} \\
\pm \frac{1}{3} \rho b \sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2} . \quad (285)
\end{aligned}$$

Equations 279 and 281 in Eqs. 273 and 274 now give, respectively,

$$\bar{B} = b\mu(\beta-\nu) \left\{ \rho a K - \frac{4(r_2+a)}{(r_2-a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) - \left(\frac{z}{z_1} \right)^2 \right] \right\} , \quad (286)$$

$$\begin{aligned}
\bar{C} = b \left\{ \rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2T}{z_1^2} \right\} \\
\pm \rho a \sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2} . \quad (287)
\end{aligned}$$

Differentiating with respect to b the condition Eq. 283 becomes

$$\int_{-z_1}^{z_1} \left(-2\bar{A} \frac{d\bar{A}}{db} + \bar{B} \frac{d\bar{B}}{db} + \bar{C} \frac{d\bar{C}}{db} \right) dz = 0 . \quad (288)$$

Here \bar{A} , \bar{B} , and \bar{C} are given by Eqs. 285, 286, and 287, respectively; and ξ is constant. Equation 288 determines b as a function of ξ . Noting that \bar{A} does not contain z Eq. 288 becomes

$$\begin{aligned}
 & - 4z_1 \left[b^2 \left\{ \frac{4}{15} \rho \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2}{3} \frac{T}{az_1^2} \right\} \right. \\
 & \left. + \frac{1}{3} \rho b \sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2} \right] \\
 & \times \left[2b \left\{ \frac{4}{15} \rho \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2}{3} \frac{T}{az_1^2} \right\} \right. \\
 & \left. + \frac{1}{3} \rho \left\{ \sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2} - \frac{b^2 K^2 \mu^2 (\beta-\nu)^2}{\sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2}} \right\} \right] \quad (289) \\
 & + b \mu^2 (\beta-\nu)^2 \int_{-z_1}^{z_1} \left\{ \rho a K - \frac{4(r_2+a)}{(r_2-a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right. \right. \\
 & \left. \left. - \left(\frac{z}{z_1} \right)^2 \right] \right\} dz + \int_{-z_1}^{z_1} \left[b \left\{ \rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2T}{z_1^2} \right\} \right. \right. \\
 & \left. \left. + \rho a \sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2} \left[\rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] \right. \right. \right. \\
 & \left. \left. - \frac{2T}{z_1^2} + \frac{\rho a b K^2 \mu^2 (\beta-\nu)^2}{\sqrt{\beta^4 \xi^2 - K^2 \mu^2 (\beta-\nu)^2 b^2}} \right] dz = 0.
 \end{aligned}$$

This expression determines b as a function of ξ . Let the Maclaurin series expansion of this function be

$$b = b_1 \xi + b_3 \xi^3 + b_5 \xi^5 + \dots \quad (290)$$

wherein the coefficients b_1, b_3, \dots are unknown constants, and the terms containing even powers of ξ are omitted because b is evidently an odd function of ξ . Good through second-order terms in ξ Eq. 290 becomes

$$b \cong b_1 \xi. \quad (291)$$

Substituting in Eq. 289 we see that the first term is of third order in ξ , whereas the other two terms are of first order. We therefore omit the first term, the resulting expression being good through second-order terms in ξ . Dividing this by ξ we obtain the following relation for determining b_1 .

$$\begin{aligned} & b_1 \mu^2 (\beta - \nu)^2 \int_{-z_1}^{z_1} \left\{ \rho a K - \frac{4(r_2 + a)}{(r_2 - a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2 - a}{r_2 + a} \right) - \left(\frac{z}{z_1} \right)^2 \right] \right\}^2 dz \\ & + \int_{-z_1}^{z_1} \left[b_1 \left\{ \rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \left[\frac{2a(\beta - \nu)^2}{r_2 - a} - \beta^2 \right] - \frac{2T}{z_1^2} \right\} \right. \\ & \left. \pm \rho a \sqrt{\beta^4 - K^2 \mu^2 (\beta - \nu)^2 b_1^2} \right] \left[\rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \right. \\ & \left. \times \left[\frac{2a(\beta - \nu)^2}{r_2 - a} - \beta^2 \right] - \frac{2T}{z_1^2} + \frac{\rho a K^2 \mu^2 (\beta - \nu)^2 b_1}{\sqrt{\beta^4 - K^2 \mu^2 (\beta - \nu)^2 b_1^2}} \right] dz = 0. \end{aligned} \quad (292)$$

Equations 279 and 281 now become

$$\beta^2 \sin \phi = K \mu (\beta - \nu) b_1, \quad (293)$$

$$\beta^2 \cos \phi = \pm \sqrt{\beta^4 - K^2 \mu^2 (\beta - \nu)^2 b_1^2}, \quad (294)$$

which substituted in Eq. 292 give

$$\begin{aligned} & \frac{\beta^2}{K} \mu (\beta - \nu) \sin \phi \left[2z_1 \left\{ \rho a K - \frac{4(r_2 + a)}{(r_2 - a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2 - a}{r_2 + a} \right) \right] \right\}^2 \right. \\ & \left. + \frac{16(r_2 + a)z_1}{3(r_2 - a)^2} \left\{ \rho a K - \frac{4(r_2 + a)}{(r_2 - a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2 - a}{r_2 + a} \right) \right] \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{32(r_2+a)^2 z_1}{5(r_2-a)^4} \Big] + \int_{-z_1}^{z_1} \left[\frac{\beta^2 \sin \phi}{K\mu(\beta-\nu)} \left\{ \rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \right. \right. \\
 & \times \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2T}{z_1^2} \Big\} + \rho a \beta^2 \cos \phi \Big] \left[\rho a \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \right. \\
 & \times \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - \frac{2T}{z_1^2} - \rho a K\mu(\beta-\nu) \tan \phi \Big] dz = 0.
 \end{aligned} \tag{295}$$

Here ϕ replaces b_1 as the unknown quantity, the two being related by Eq. 293. The integral in Eq. 295 is

$$\begin{aligned}
 & \frac{\beta^2 \sin \phi}{K\mu(\beta-\nu)} \rho^2 a^2 \frac{16}{15} z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right]^2 \\
 & + \rho a \frac{4}{3} z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] \left[\frac{\beta^2 \sin \phi}{K\mu(\beta-\nu)} \left(- \frac{2T}{z_1^2} \right. \right. \\
 & - \rho a K\mu(\beta-\nu) \tan \phi \Big) + \frac{\beta^2 \sin \phi}{K\mu(\beta-\nu)} \left(- \frac{2T}{z_1^2} \right) + \rho a \beta^2 \cos \phi \Big] \\
 & + 2z_1 \left[\frac{\beta^2 \sin \phi}{K\mu(\beta-\nu)} \left(- \frac{2T}{z_1^2} \right) + \rho a \beta^2 \cos \phi \right] \left[- \frac{2T}{z_1^2} \right. \\
 & - \rho a K\mu(\beta-\nu) \tan \phi \Big],
 \end{aligned} \tag{296}$$

which in Eq. 295 gives

$$\bar{P} \sin \phi + \bar{Q} \cos \phi + \bar{R} \sin \phi \tan \phi = 0 \tag{297}$$

where

$$\bar{P} = \frac{\beta^2 \mu}{K} (\beta-\nu) \left[2z_1 \left\{ \rho a K - \frac{4(r_2+a)}{(r_2-a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right] \right\} \right.$$

$$\begin{aligned}
& + \frac{16(r_2+a)z_1}{3(r_2-a)^2} \left\{ \rho a K - \frac{4(r_2+a)}{(r_2-a)^2} \left[1 + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right] \right\} \\
& + \frac{32(r_2+a)^2 z_1}{5(r_2-a)^4} + \frac{\beta^2 \rho^2 a^2}{K\mu(\beta-\nu)} \frac{16}{15} z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right]^2 \\
& + \rho a \frac{4}{3} z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] \frac{\beta^2}{K\mu(\beta-\nu)} \left(-\frac{4T}{z_1^2} \right) \\
& + 2z_1 \frac{\beta^2}{K\mu(\beta-\nu)} \left(-\frac{2T}{z_1^2} \right)^2 - 2z_1 \rho^2 a^2 \beta^2 K\mu(\beta-\nu) \\
\bar{P} = & \frac{2z_1 \beta^2 \mu}{K} (\beta-\nu) \left[\left\{ \right\}^2 + \frac{8(r_2+a)}{3(r_2-a)^2} \left\{ \right\} + \frac{16(r_2+a)^2}{9(r_2-a)^4} \right. \\
& + \frac{(r_2+a)^2}{(r_2-a)^4} \left(\frac{16}{5} - \frac{16}{9} \right) \left. \right] + \frac{16\beta^2 z_1}{15K\mu(\beta-\nu)} \left\{ \rho^2 a^2 \left[\right]^2 \right. \\
& - 5 \frac{T}{z_1^2} \rho a \left[\right] + \frac{25T^2}{4z_1^4} + \frac{T^2}{z_1^4} \left(\frac{15}{2} - \frac{25}{4} \right) \left. \right\} - 2z_1 \rho^2 a^2 \beta^2 K\mu(\beta-\nu) \\
\bar{P} = & \frac{2z_1 \beta^2 \mu}{K} (\beta-\nu) \left\{ \left(\rho a K - \frac{4(r_2+a)}{(r_2-a)^2} \left[\frac{2}{3} + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right] \right)^2 \right. \\
& + \frac{64(r_2+a)^2}{45(r_2-a)^4} \left. \right\} + \frac{16\beta^2 z_1}{15K\mu(\beta-\nu)} \left\{ \left(\rho a \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] \right. \right. \\
& - \frac{5T}{2z_1^2} \left. \right)^2 + \frac{5T^2}{4z_1^4} \left. \right\} - 2z_1 \rho^2 a^2 \beta^2 K\mu(\beta-\nu) \\
\bar{P} = & \frac{16z_1 \beta^2 \mu}{K} (\beta-\nu) \left(\frac{(r_2+a)}{(r_2-a)^2} \left[\frac{2}{3} + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right] \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ 2 \frac{(r_2+a)}{(r_2-a)^2} \left[\frac{2}{3} + \left(\frac{a}{z_1} \right)^2 \left(\frac{r_2-a}{r_2+a} \right) \right] - \rho a K \right\} \\
& + \frac{8(r_2+a)^2}{45(r_2-a)^4} + \frac{16z_1\beta^2}{15K\mu(\beta-\nu)} \left\{ \left(\rho a \left[\frac{2a(\beta-\nu)^2}{r_2-a} \right. \right. \right. \\
& \left. \left. \left. - \beta^2 \right] - \frac{5T}{2z_1^2} \right)^2 + \frac{5T^2}{4z_1^4} \right\},
\end{aligned} \tag{298}$$

$$\bar{Q} = \frac{4}{3} \rho^2 a^2 \beta^2 z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] - 4\rho a \beta^2 \frac{T}{z_1}, \tag{299}$$

$$\bar{R} = - \frac{4}{3} \rho^2 a^2 \beta^2 z_1 \left[\frac{2a(\beta-\nu)^2}{r_2-a} - \beta^2 \right] + 4\rho a \beta^2 \frac{T}{z_1}. \tag{300}$$

We now see that

$$\bar{R} = - \bar{Q}; \tag{301}$$

hence Eq. 297 can be written

$$\bar{P} \sin\phi + \bar{Q}(\cos\phi - \sin\phi \tan\phi) = 0,$$

or

$$\frac{\bar{P} \sin\phi \cos\phi + \bar{Q}(\cos^2\phi - \sin^2\phi)}{\cos\phi} = 0,$$

$$\left(\frac{1}{2} \bar{P} \sin 2\phi + \bar{Q} \cos 2\phi \right) \sec\phi = 0, \tag{302}$$

$$\tan 2\phi = - \frac{2\bar{Q}}{\bar{P}}. \tag{303}$$

Since the tangent has a period π we see that this equation has a root such that

$$0 \leq 2\phi < \pi,$$

or

$$0 \leq \phi < \frac{\pi}{2} . \quad (304)$$

Other roots of Eq. 303 can be obtained by adding multiples of π to 2ϕ , or multiples of $\frac{\pi}{2}$ to ϕ . We thus obtain four values of ϕ which satisfy Eq. 303 and are equally spaced over the complete circle 2π . Of these we retain only those two which are smaller than π , as is evident from the figure on page 12 and the fact that y_0 in Eq. 39 must be negative. These two values differ by $\frac{\pi}{2}$, and of them we wish to choose that one which minimizes Eq. 283, or, what is the same thing since $\bar{A} \ll \bar{B}, \bar{C}$, choose that for which

$$I = \int_{-z_1}^{z_1} (\bar{B}^2 + \bar{C}^2) dz = \text{minimum}. \quad (305)$$

That $\bar{A} \ll \bar{B}, \bar{C}$ follows from Eqs. 285, 286, 287, and 291, from which we see that \bar{A} is of second order in ξ , whereas \bar{B} and \bar{C} are of first order. Noting Eq. 291 and the fact that ξ is arbitrarily small but fixed, we see that the variable which we determine in minimizing Eq. 305 is b_1 , which in view of Eq. 293 is a measure of ϕ . The left-hand side of Eq. 297 and hence Eq. 302 was obtained by dividing $\frac{dI}{db}$ by ξ , thus

$$\frac{1}{\xi} \frac{dI}{db} = \frac{1}{\xi^2} \frac{dI}{db_1} = \left(\frac{1}{2} \bar{P} \sin 2\phi + \bar{Q} \cos 2\phi \right) \sec \phi. \quad (306)$$

Noting Eq. 293 it follows that

$$\begin{aligned} \frac{1}{\xi^2} \frac{d^2 I}{db_1^2} &= \frac{1}{\xi^2} \frac{\frac{d}{d\phi} \left(\frac{dI}{db_1} \right)}{\frac{db_1}{d\phi}} = \frac{\kappa \mu (\beta - \nu)}{\beta^2 \cos \phi} \left\{ (\bar{P} \cos 2\phi - 2\bar{Q} \sin 2\phi) \sec \phi \right. \\ &\quad \left. + \left(\frac{1}{2} \bar{P} \sin 2\phi + \bar{Q} \cos 2\phi \right) \sec \phi \tan \phi \right\}. \end{aligned} \quad (307)$$

The two values of ϕ which we have retained satisfy Eq. 302. For each of these Eq. 307 becomes

$$\frac{d^2 I}{db_1^2} = \frac{K\mu(\beta-\nu)\xi^2}{\beta^2 \cos^2 \phi} (\bar{P}\cos 2\phi - 2\bar{Q}\sin 2\phi). \quad (308)$$

Since $\beta > \nu$ and K is positive due to Eq. 279, the sign of this expression is that of

$$\bar{P} \cos 2\phi - 2\bar{Q} \sin 2\phi. \quad (309)$$

Since our two values of ϕ differ by $\frac{\pi}{2}$ we see that Eq. 309 has the same absolute value for both, but opposite signs. We choose that value of ϕ for which this sign is positive, corresponding to a minimum of I (for fixed ξ), and discard the other value of ϕ , which corresponds to a maximum.

We see from Eq. 303 that the two possible quadrants of 2ϕ , and hence the two corresponding 45-deg sectors of ϕ , are determined by the sign of $\frac{\bar{Q}}{\bar{P}}$. For each of these the signs of $\sin 2\phi$ and $\cos 2\phi$ are also determined. It follows that for each of these the sign of Eq. 309 is determined by the signs of \bar{P} and \bar{Q} . Given the signs of \bar{P} and \bar{Q} we can hence see immediately which 45-deg sector must contain ϕ in order that Eq. 303 be satisfied and Eq. 309 be positive. These sectors are as follows.

| Sign of \bar{P} | Sign of \bar{Q} | Sector containing ϕ |
|-------------------|-------------------|---|
| + | - | $0 < \phi < \frac{\pi}{4}$ |
| - | - | $\frac{\pi}{4} < \phi < \frac{\pi}{2}$ |
| - | + | $\frac{\pi}{2} < \phi < \frac{3\pi}{4}$ |
| + | + | $\frac{3\pi}{4} < \phi < \pi$ |

In any particular case once the signs of \bar{P} and \bar{Q} have been determined, this table specifies the sector which contains ϕ .

DETERMINATION OF THE DAMPING CONSTANT AND PRECESSIONAL ANGULAR VELOCITY

Now that we have determined ϕ , b follows from Eq. 279, after which \bar{x} follows from Eq. 93. Noting that the xy -axes of the figure on page 31, have their origin at S and are rotated clockwise by an angle ϕ from those in the figure on page 12, whose origin is at W , as indicated in the figure on page 66, we see that x_0 and y_0 on page 12 are given by

$$\begin{aligned} x_0 &= \xi + \bar{x} \cos\phi = \xi + \frac{2ba^2 \cos\phi}{3(r_2^2 - a^2)} \\ &= \left(1 + \frac{2a^2 \beta^2 \sin\phi \cos\phi}{3(r_2^2 - a^2) K\mu(\beta - \nu)}\right) \xi, \end{aligned} \quad (310)$$

$$\begin{aligned} y_0 &= -\bar{x} \sin\phi = -\frac{2ba^2 \sin\phi}{3(r_2^2 - a^2)} \\ &= -\left(\frac{2a^2 \beta^2 \sin^2\phi}{3(r_2^2 - a^2) K\mu(\beta - \nu)}\right) \xi. \end{aligned}$$

These are the coordinates of the center of gravity of the mercury. The components of the force exerted by a damper on the main spinning body now follow from Eq. 30, thus

$$\begin{aligned} |F_1| &= -My_0 \beta^2 = M\beta^2 \left(\frac{2a^2 \beta^2 \sin^2\phi}{3(r_2^2 - a^2) K\mu(\beta - \nu)}\right) \xi, \\ |F_2| &= Mx_0 \beta^2 = M\beta^2 \left(1 + \frac{2a^2 \beta^2 \sin\phi \cos\phi}{3(r_2^2 - a^2) K\mu(\beta - \nu)}\right) \xi; \end{aligned} \quad (311)$$

also the torque components exerted by a pair of dampers on the main spinning body are given by Eqs. 32, 34, and 311, thus

$$\begin{aligned} x \text{ component of torque} &= k_x \xi, \\ y \text{ component of torque} &= k_y \xi, \\ z \text{ component of torque} &= k_z \xi^2 \end{aligned} \quad (312)$$

where

$$\begin{aligned}k_x &= M\beta^2 Lk \sin\phi, \\k_y &= M\beta^2 L(2 + k \cos\phi), \\k_z &= M\beta^2 k \sin\phi = \frac{k_x}{L},\end{aligned}\tag{313}$$

and

$$k = \frac{4a^2\beta^2 \sin\phi}{3(r_2^2 - a^2)K\mu(\beta - \nu)}.\tag{314}$$

We note that the coordinates in Eq. 310, the force components in Eq. 311, and the x and y torque components in Eq. 312 are all, closely, proportional to ξ , since in these expressions the coefficients of ξ depend upon β and ν , which vary but little. The z component of torque, however, varies as ξ^2 .

From Eq. 312 we now have

$$\dot{\psi} = \frac{k_z}{A} \xi^2;\tag{315}$$

hence good through first-order terms in ξ , ν is constant.

The results given by the approximate picture presented in the last part of the section between pages 7 and 14 now follow from Eqs. 35 and 37, which give

$$-\dot{\xi} = \frac{k_x L}{A\nu} \xi,\tag{316}$$

$$\begin{array}{l} \text{Reduction in preces-} \\ \text{sional angular velo-} \\ \text{city} \end{array} = \frac{A\nu}{B} - \beta = \frac{k_y L}{A\nu}.\tag{317}$$

Here Eq. 317 can be used to determine β . Starting with the approximate value $\beta \approx \frac{A\nu}{B}$ we determine the right-hand side of Eq. 317, which is the amount by which $\frac{A\nu}{B}$ must be reduced to give the next approximation to β . Repeating this process

using the corrected value of β we obtain the next approximation to β --and so on.

In Eq. 316 the coefficient of ξ is essentially constant; hence ξ is, closely, a dying exponential, the damping factor (coefficient of t in the exponential) being

$$\alpha = - \frac{k L}{A\nu}. \quad (318)$$

The negative reciprocal of α is the time constant.

The results given by the approximate picture presented previously (pp. 25-28) follow from Eqs. 73 and 74, thus, noting Eq. 312,

$$\frac{d}{dt} (\xi\beta) = - \frac{k L}{B} \xi,$$

or, if β is considered to be essentially constant,

$$\dot{\xi} = - \frac{k L}{B\beta} \xi. \quad (319)$$

Also

$$\frac{\xi\beta}{L} (A\nu - B\beta) = k_y \xi,$$

or

$$\begin{array}{l} \text{Reduction in preces-} \\ \text{sional angular velo-} \\ \text{city} \end{array} = \frac{A\nu}{B} - \beta = \frac{k L}{B\beta}. \quad (320)$$

The damping factor corresponding to Eq. 319 is

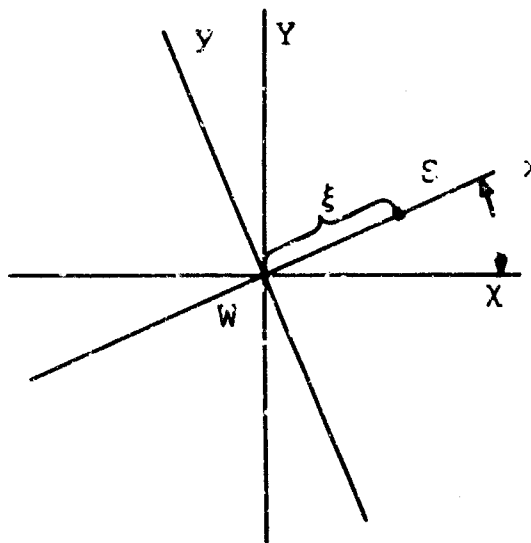
$$\alpha = - \frac{k L}{B\beta}. \quad (321)$$

Equation 317 differs from Eq. 320, and Eq. 318 differs from Eq. 321 in that $A\nu$ appears in place of $B\beta$; however, the ratio of these two quantities should approximate unity, since

$\frac{A\dot{\nu}}{B}$ approximates β . It follows that the results given by the two pictures are in approximate agreement. It may be noted that in Eqs. 319, 320, and 321 β is the angular velocity seen by an observer who is located at the instantaneous center of curvature of the path of S.

APPLICATION OF PROCEDURES FOR LINEAR DIFFERENTIAL EQUATIONS

Since the coefficients of ξ in the first two equations of Eq. 312 are closely constant it appears that we can check and perhaps improve the treatment in the section on pages 25-28 by using the methods available for solving systems of linear differential equations. Let us choose a stationary set of XY axes in the plane of the upper damper with its origin at W, in the vicinity of S, as shown in the figure below. $\xi = WS$, and is hence the x coordinate of S. Knowing



the torque components along the x and y axes from Eq. 312, we can obtain those along the X and Y axes immediately, thus

$$\begin{aligned}
 \text{X component of torque} &= k_x \xi \cos \theta - k_y \xi \sin \theta \\
 &= k_x X - k_y Y, \\
 \text{Y component of torque} &= k_x \xi \sin \theta + k_y \xi \cos \theta \\
 &= k_x Y + k_y X.
 \end{aligned} \tag{322}$$

The differential equations for the motion of S are hence, closely,

$$\begin{aligned} \frac{B}{L} \ddot{X} + \frac{A\nu}{L} \dot{Y} &= k_x Y + k_y X \\ \frac{A\nu}{L} \dot{X} - \frac{B}{L} \ddot{Y} &= k_x X - k_y Y, \end{aligned} \quad (323)$$

or, denoting $\frac{d}{dt}$ by D,

$$\begin{aligned} (BD^2 - Lk_y)X + (A\nu D - Lk_x)Y &= 0 \\ (A\nu D - Lk_x)X - (BD^2 - Lk_y)Y &= 0. \end{aligned} \quad (324)$$

Equating the determinant of the system to zero we obtain

$$\begin{vmatrix} (BD^2 - Lk_y) & (A\nu D - Lk_x) \\ (A\nu D - Lk_x) & -(BD^2 - Lk_y) \end{vmatrix} = 0, \quad (325)$$

$$\begin{aligned} -(BD^2 - Lk_y)^2 - (A\nu D - Lk_x)^2 &= 0, \\ BD^2 - Lk_y &= \pm i(A\nu D - Lk_x). \end{aligned} \quad (326)$$

Equation 325 is a quartic polynomial in D, whose four roots are composed of the two pairs of roots of Eq. 326 that are obtained for the two possible signs. Since it is evident that the conjugates of the two roots obtained for one sign in Eq. 326 are roots of this equation with the opposite choice of sign, it follows that we need consider but one sign. Choosing the plus sign Eq. 326 may be written

$$BD^2 - iA\nu D - L(k_y - ik_x) = 0; \quad (327)$$

hence

$$D = \frac{1}{2B} \left[iA\nu \pm \sqrt{-A^2\nu^2 + 4BL(k_y - ik_x)} \right],$$

$$D = \frac{iA\nu}{2B} \left[1 \pm \sqrt{1 - \frac{4BL}{A^2\nu^2} (k_y - ik_x)} \right]. \quad (328)$$

For either choice of sign we obtain a root, whose conjugate is also a root of Eq. 325, thus

$$D = \alpha \pm i\beta. \quad (329)$$

Corresponding to such a pair of roots we have a mode, for which the damping constant is α , and the precessional angular velocity is β ($\beta > 0$).

α and β can be obtained exactly from Eq. 328; however, by applying the binomial theorem to the radical in Eq. 328 we can obtain the following approximate expression

$$\begin{aligned} D &= \frac{iA\nu}{2B} \left\{ 1 \pm \left[1^{\frac{1}{2}} + \frac{1}{2} 1^{-\frac{1}{2}} \left(-\frac{4BL}{A^2\nu^2} (k_y - ik_x) \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left(-\frac{1}{2} \right) \frac{1}{2} \left(-\frac{4BL}{A^2\nu^2} (k_y - ik_x) \right)^2 + \dots \right] \right\} \\ D &= \frac{iA\nu}{2B} \left\{ 1 \pm \left[1 - \frac{2BL}{A^2\nu^2} (k_y - ik_x) \right. \right. \\ &\quad \left. \left. - \frac{2B^2L^2}{A^4\nu^4} (k_y^2 - k_x^2 - 12k_x k_y) + \dots \right] \right\}. \end{aligned} \quad (330)$$

Noting Eq. 329 this gives for the plus sign

$$\begin{aligned} D &= -\frac{Lk_x}{A\nu} - \frac{2BL^2 k_x k_y}{A^3\nu^3} + \dots \\ &+ i \left[\frac{A\nu}{B} - \frac{Lk_y}{A\nu} - \frac{BL^2}{A^3\nu^3} (k_y^2 - k_x^2) + \dots \right], \end{aligned} \quad (331)$$

whence

$$\alpha = -\frac{Lk_x}{A\nu} - \frac{2BL^2 k_x k_y}{A^3\nu^3} + \dots, \quad (332)$$

$$\beta = \frac{A\nu}{B} - \frac{Lk_y}{A\nu} - \frac{BL^3}{A^3\nu^3} (k_y^2 - k_x^2) + \dots$$

We note that through first-order terms in k_x and k_y Eq. 332 agrees with Eqs. 317 and 318.

The corresponding mode can be obtained using complex notation for damped sinusoids; thus if \bar{X} and \bar{Y} denote the complex numbers corresponding to X and Y , respectively, and $D = \alpha + i\beta$ the differential equations (Eq. 324) become⁷

$$\begin{aligned} (BD^2 - Lk_y)\bar{X} + (A\nu D - Lk_x)\bar{Y} &= 0 \\ (A\nu D - Lk_x)\bar{X} - (BD^2 - Lk_y)\bar{Y} &= 0. \end{aligned} \quad (333)$$

The solution of this set of linear, homogeneous, algebraic equations can be written down immediately by noting that \bar{X} and \bar{Y} are proportional to the cofactors of the elements of either row of the determinant in Eq. 325--say the first, thus

$$\begin{aligned} \bar{X} &= -c_1 (BD^2 - Lk_y), \\ \bar{Y} &= -c_1 (A\nu D - Lk_x) \end{aligned} \quad (334)$$

⁷ Complex notation for damped sinusoids can easily be obtained as follows. Let a one-to-one correspondence between damped sinusoids of fixed damping constant α and angular frequency ω , and points of the complex plane be defined by the relation

$$Ae^{\alpha t} \sin(\omega t + \theta) \sim A \angle \theta = A(\cos \theta + i \sin \theta);$$

then it can readily be shown that the complex number corresponding to a linear combination of damped sinusoids, all having the same α and ω , is equal to that same linear combination of the complex numbers corresponding to the component damped sinusoids; also that the complex number corresponding to the time derivative of a damped sinusoid is equal to $(\alpha + i\omega)$ times the complex number corresponding to the damped sinusoid itself.

where c_1 is an arbitrary complex constant. Since Eq. 326 with the plus sign is valid Eq. 334 becomes

$$\begin{aligned}\bar{X} &= ic, \\ \bar{Y} &= c\end{aligned}\tag{335}$$

where $c = |c| \angle \theta_c$ is an arbitrary complex constant. It follows that

$$\begin{aligned}X &= |c| e^{\alpha t} \cos(\beta t + \theta_c), \\ Y &= |c| e^{\alpha t} \sin(\beta t + \theta_c);\end{aligned}\tag{336}$$

also

$$\xi = \sqrt{X^2 + Y^2} = |c| e^{\alpha t}.\tag{337}$$

The phase of the damped sinusoids Eq. 336 is hence such that S spirals around W with an exponentially decreasing radius.

It might appear that another mode can be obtained by using the minus sign in Eq. 328, and that the motion of S is a composite of the two modes; however, for the torque equations (Eq. 312) to be valid it is necessary that the origin be at W, on the precession axis. Furthermore with the minus sign the value of β obtained approximates $\frac{Lk_y}{A\nu}$, as can be seen from Eq. 330. At this extremely low precessional angular velocity the above analysis of the action of the mercury damper, upon which Eq. 312 is based, is no longer valid.

POLAR FORM OF THE TORQUE DIFFERENTIAL EQUATIONS

Referring to the figure on page 98 we note in regard to the motion of S that

$$\begin{aligned}\text{Radial component of velocity} &= \dot{\xi}, \\ \text{Transverse component of velocity} &= \xi \dot{\theta}, \\ \text{Radial component of acceleration} &= \ddot{\xi} - \xi \dot{\theta}^2, \\ \text{Transverse component of acceleration} &= 2\dot{\xi}\dot{\theta} + \xi \ddot{\theta}.\end{aligned}\tag{338}$$

Noting that $\beta = \dot{\theta}$ it follows that

$$\begin{aligned}\frac{A\nu}{L} \dot{\xi} - \frac{B}{L} (2\beta\dot{\xi} + \xi\dot{\beta}) &= 2\tau_x \\ \frac{B}{L} (\ddot{\xi} - \xi\beta^2) + \frac{A\nu}{L} \xi\beta &= 2\tau_y,\end{aligned}$$

closely where $2\tau_x$ and $2\tau_y$ are the x and y components, respectively, of the total torque exerted by both dampers on the main spinning body. These relations may be written

$$\begin{aligned}(A\nu - 2B\beta)\dot{\xi} - B\xi\dot{\beta} &= 2L\tau_x \\ B\ddot{\xi} + (A\nu - B\beta)\beta\xi &= 2L\tau_y.\end{aligned}\tag{339}$$

Applying Eq. 339 to the case just considered in the previous section we have, noting Eqs. 312 and 337,

$$\dot{\xi} = \alpha\xi, \quad \ddot{\xi} = \alpha^2\xi, \quad \dot{\beta} = 0, \quad 2\tau_x = k_x\xi, \quad 2\tau_y = k_y\xi; \tag{340}$$

hence

$$\begin{aligned}(A\nu - 2B\beta)\alpha &= Lk_x \\ B\alpha^2 + (A\nu - B\beta)\beta &= Lk_y.\end{aligned}\tag{341}$$

These relations may be written

$$\alpha = \frac{Lk_x}{A\nu - 2B\beta}, \tag{342}$$

$$\frac{A\nu}{B} - \beta = \frac{Lk_y}{B\beta} - \frac{\alpha^2}{\beta}, \tag{343}$$

which give the damping constant and the reduction in the precessional angular velocity. Again we note that since $\frac{A\nu}{B}$ approximates β , Eq. 342 is in approximate agreement with Eqs. 318 and 321. In using Eqs. 342 and 343 β is first assigned the approximate value $\frac{A\nu}{B}$, and a first approximation to α calculated from Eq. 342. This and the approximate value of

β placed in the right-hand side of Eq. 343 give the amount by which $\frac{A\nu}{B}$ must be reduced in order to obtain the second approximation to β . Using this instead of $\frac{A\nu}{B}$ we can now repeat the above process to get a second approximation to α , a third approximation to β , and so on.

Finally we note that the equations (Eq. 339) provide a better means of determining damping behavior and precessional angular velocity than does the method described in the section on pages 25-28. We shall see that these equations will prove useful in considering the case of the crescent configuration of the mercury, for which the torque components are given by expressions which differ greatly from Eq. 312, and for which therefore linear differential equation procedures cannot be used.

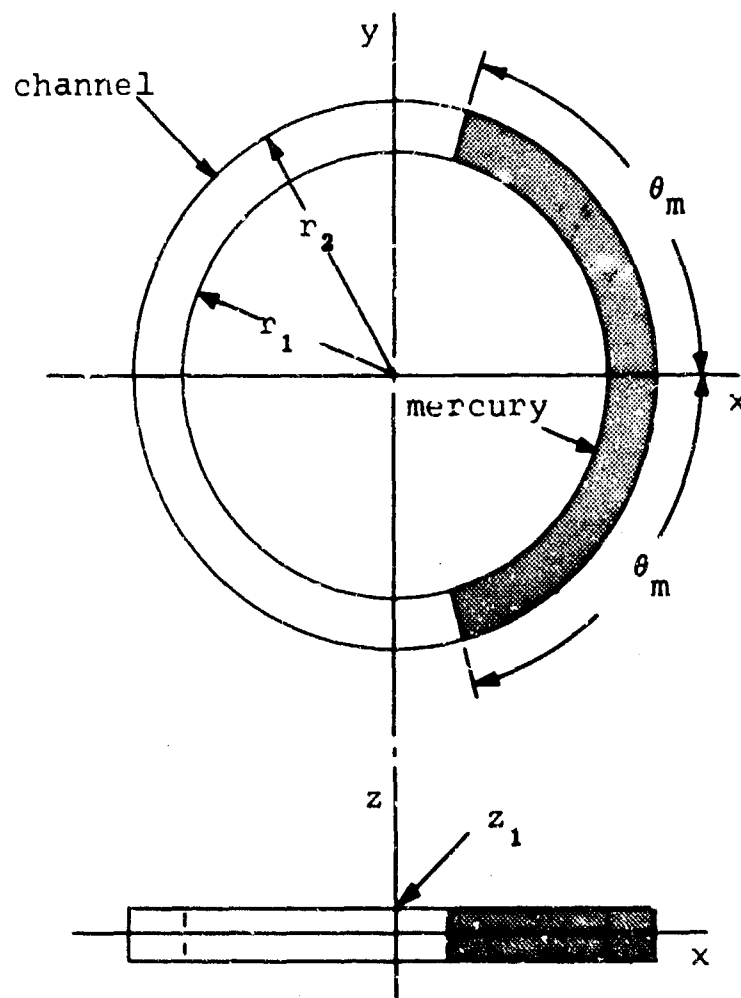
CASE WHERE THE MERCURY CONFIGURATION IS CRESCENT-SHAPED

We shall next consider the case where the mercury is crescent-shaped, and completely fills an angle $2\theta_m$ of the channel, or race, the flow being turbulent. In so doing we shall use the same notation which we have been using in connection with the eccentric annular configuration. Instead of the figure on page 31 we now have the configuration shown in the figure on the following page. The volume of the mercury is evidently

$$\text{Volume of mercury} = \left(\frac{2\theta_m}{2\pi} \right) 2z_1 \pi (r_2^2 - r_1^2) = 2z_1 \theta_m (r_2^2 - r_1^2). \quad (344)$$

Again denoting the x coordinate of the center of gravity of the mercury by \bar{x} we have

$$\begin{aligned} \bar{x} \text{ times mercury volume} &= \int_{-\theta_m}^{\theta_m} \int_{r_1}^{r_2} \int_{-z_1}^{z_1} r^3 \cos \theta d\theta dr dz \\ &= \frac{4z_1}{3} (r_2^3 - r_1^3) \sin \theta_m; \end{aligned}$$

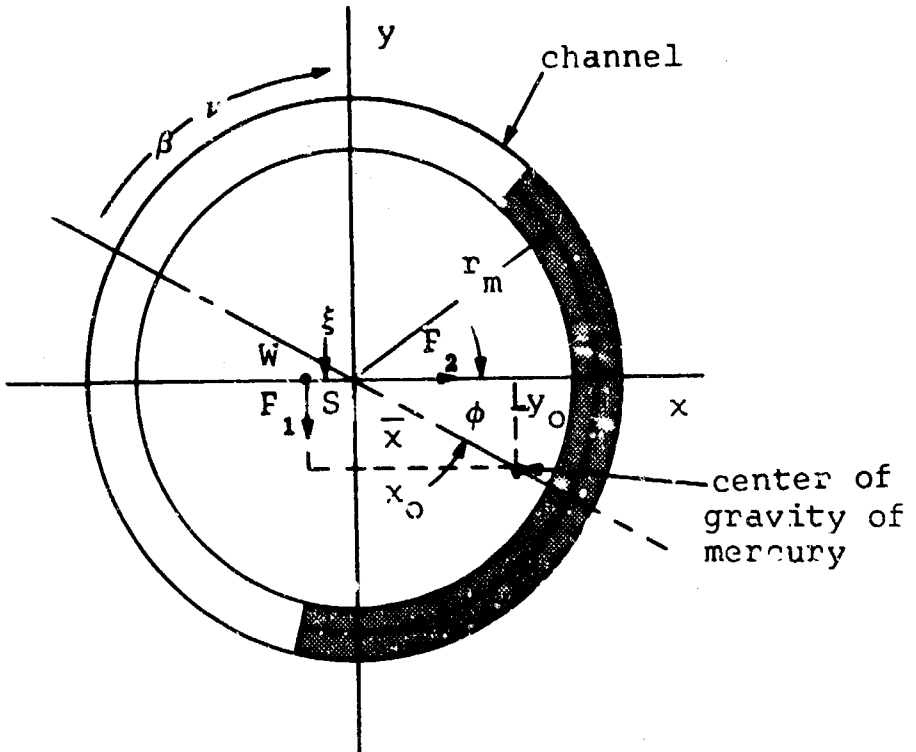


hence dividing by Eq. 344

$$\bar{x} = \frac{2}{3} \left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} \right) \left(\frac{\sin \theta_m}{\theta_m} \right), \quad (345)$$

or

$$\bar{x} = \frac{2}{3} \left(\frac{r_2^3 + r_1 r_2 + r_1^2}{r_2 + r_1} \right) \left(\frac{\sin \theta_m}{\theta_m} \right). \quad (346)$$



Here θ_m is determined by Eq. 344, since the mercury volume is known.

If, as before, we observe the system while riding with axes which rotate counterclockwise at the precessional angular velocity β we see the configuration shown in the above figure. Noting the figure on page 12 we now take moments about S, as described in the section on pages 15 and 16, and thus obtain

$$|F_1|_t = \kappa_0 r_m^3 (\beta - \nu)^2. \quad (347)$$

Here r_m is the mean radius of the mercury, and $r_m(\beta - \nu)$ is the velocity of the mercury relative to the channel. K_0 is a coefficient obtained from Chezy's formula, which coefficient depends upon the shape of the channel, the area of the surface of contact of the mercury with the channel, and the density and viscosity of the mercury.⁸

¹ Rouse, Hunter, and J. W. Howe. Basic Mechanics of Fluids. New York, Wiley, 1953. P. 149.

Noting Eqs. 32, 34, and the fact that there are two dampers we obtain the following components of the torque exerted by the mercury on the main spinning body.

$$\begin{aligned} \text{Total x component of torque} &= 2\tau_x = 2|F_1|L \\ &= \frac{2LK_0 r_m^3 (\beta - \nu)^2}{\xi}, \end{aligned} \quad (348)$$

$$\begin{aligned} \text{Total z component of torque} &= 2\tau_z = 2|F_1|\xi \\ &= 2K_0 r_m^3 (\beta - \nu)^2. \end{aligned} \quad (349)$$

Equation 347 in the first equation of Eq. 30 now gives

$$y_0 = - \frac{K_0 r_m^3 (\beta - \nu)^2}{M\beta^2 \xi}, \quad (350)$$

whence, noting Eq. 346 and the figure on page 106,

$$\sin\phi = - \frac{y_0}{\bar{x}}, \quad (351)$$

which relation determines ϕ . Again noting the figure on page 106 we now have

$$x_0 = \xi + \bar{x} \cos\phi, \quad (352)$$

whence the second relation in Eq. 30 gives

$$|F_2| = M\beta^2 (\xi + \bar{x} \cos\phi). \quad (353)$$

It now follows from the second equation of Eq. 34 that for the two dampers together

$$\begin{aligned} \text{Total y component of torque} &= 2\tau_y = 2|F_2|L \\ &= 2LM\beta^2 (\xi + \bar{x} \cos\phi). \end{aligned} \quad (354)$$

Equating Eq. 348 to $A\dot{\nu}$ we obtain for the rate of increase of the angular velocity of spin

$$\dot{\nu} = \frac{2K_0 r_m^3 (\beta - \nu)^2}{A}. \quad (355)$$

Finally let us apply Eq. 339 to the present case in order to obtain information on the damping and the decrease in precessional angular velocity. In so doing we shall neglect the variation of β and ν . Substituting Eq. 348 in the first equation of Eq. 339 and placing $\beta=0$ we obtain

$$(A\nu - 2B\beta)\dot{\xi} = \frac{2L^2 K_0 r_m^3 (\beta - \nu)^2}{\xi}. \quad (356)$$

Letting ξ_0 denote the initial value of ξ we separate variables and integrate, thus

$$(A\nu - 2B\beta) \frac{(\xi^2 - \xi_0^2)}{2} = 2L^2 K_0 r_m^3 (\beta - \nu)^2 t$$

or

$$\xi = \sqrt{\xi_0^2 - \frac{4L^2 K_0 r_m^3 (\beta - \nu)^2}{2B\beta - A\nu} t}. \quad (357)$$

Finally we substitute Eq. 354 in the second equation of Eq. 339, divide by B , and solve for the binomial $\left(\frac{A\nu}{B} - \beta\right)$. The decrease in the precessional angular velocity is thus found to be

$$\frac{A\nu}{B} - \beta = \frac{1}{B\beta\xi} [2L^2 M\beta^2 (\xi + \bar{x}\cos\phi) - B\ddot{\xi}]. \quad (358)$$

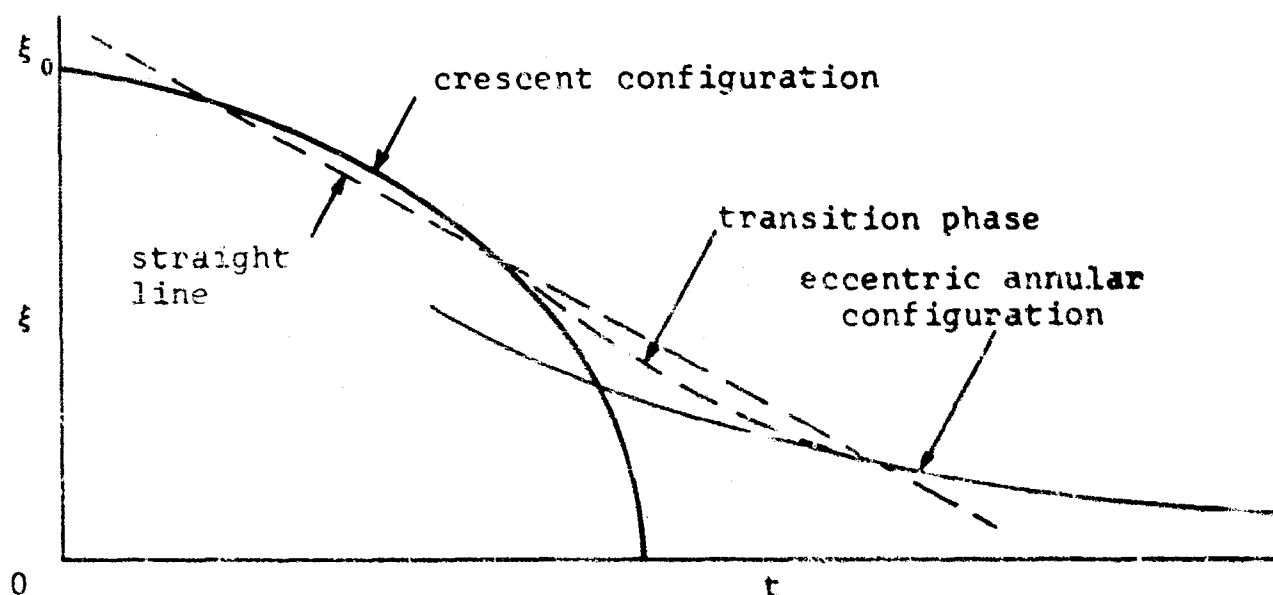
Here it is likely that in the bracket ξ and $\ddot{\xi}$ can be neglected; but this can best be decided in specific numerical cases. In this connection we note that

$$\ddot{\xi} = - \frac{1}{\xi^3} \left(\frac{2L^2 K_0 r_m^3 (\beta - \nu)^2}{2B\beta - A\nu} \right)^2, \quad (359)$$

which can be obtained by differentiating Eq. 357 twice.

Several things are immediately evident from the above analysis. First, we see from Eqs. 350 and 351 that as ξ decreases during the course of the damping ($-y_0$) and ϕ increase. The smallest value of ξ that is compatible with the crescent configuration is that for which $-y_0 = \bar{x}$, in which case $\phi = \frac{\pi}{2}$. For smaller values of ξ the mercury configuration becomes eccentric annular.

Second, we note from Eq. 357 that ξ decreases parabolically instead of exponentially, as shown below. The rate of



damping increases as ξ decreases, and is greatest just before the crescent flow breaks and goes over into the eccentric annular. In fact differentiating Eq. 357 we obtain

$$\dot{\xi} = - \frac{1}{\xi} \left(\frac{2L^2 K_0 r_m^3 (\beta - \nu)^2}{2B\beta - A\nu} \right), \quad (360)$$

from which we see that the rate of damping varies inversely as ξ , closely, instead of directly as ξ , which is the case with the eccentric annular configuration.

Finally, we see that since the first part of the curve of ξ against t is a parabola which is concave downward, and the last part of this curve is a damped exponential, which is concave upward; and since there is an intermediate part of this curve which corresponds to values of ξ for which the flow is eccentric annular but very eccentric; it follows that the ξt curve as a whole has, if differentiable, a point of inflection, and in any case should be capable of being approximated by a straight line over an appreciable range. Such a straight line is shown dotted in the figure on page 109.

Appendix A

EFFECT OF THE TILT AND THREE DIMENSIONAL NATURE
OF THE DAMPER ON THE FORCE WHICH IT EXERTS
ON THE MAIN SPINNING BODY

The force exerted by the mercury on the main spinning body was derived on pages 8-12. In this derivation the damper was considered to be essentially two dimensional, its plane being fixed, and its motion being in this plane. This simplified the derivation, and gave for the resultant force exerted by the mercury on the main spinning body a force which lies in the plane of the damper. Actually the damper is three dimensional, and is tilted slightly. This gives rise to a torque, which may shift the resultant force away from W, and which may, in addition, give rise to a torque whose vector is parallel to that of the resultant force. The purpose of this appendix is to refine the above derivation, taking into account the tilt and three dimensional nature of the damper, and see whether or not the results obtained differ appreciably from those obtained above.

Determination of F_L . Let us place a set of xyz axes with the z axis coincident with the precession axis, the +z direction being outward away from the body, and the positive x axis passing through the center of gravity of the mercury. Noting Eq. 6 it follows that

$$F_L = \int_M \beta^2 (ix + jy) dm = \beta^2 \left(i \int_M x dm + j \int_M y dm \right) \quad (A1)$$

$$F_L = i\beta^2 M \bar{x}$$

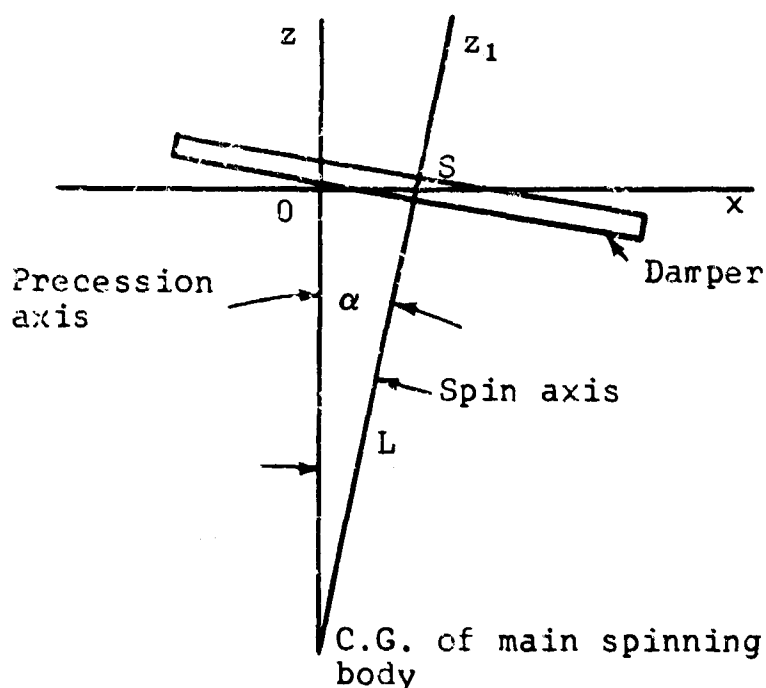
where the integrations extend throughout the mass of the mercury, and \bar{x} is the x coordinate of the center of gravity of the mercury. We thus see that the total centrifugal force is identical with that obtained by concentrating all mass at the center of gravity. This result is in agreement with Eq. 14.

Determination of F_C . Equations 15 and 16 are formally valid in the present case; hence we again obtain the result (Eq. 17), namely

$$F_C = 0. \quad (A2)$$

Since the total Coriolis force is zero the total force $-F$ is composed of F_L alone, as before.

Determination of τ_L . Let us place a set of xyz axes with the z axis coincident with the precession axis, the positive z direction being outward, and the positive x axis passing through S, as shown in figure below.

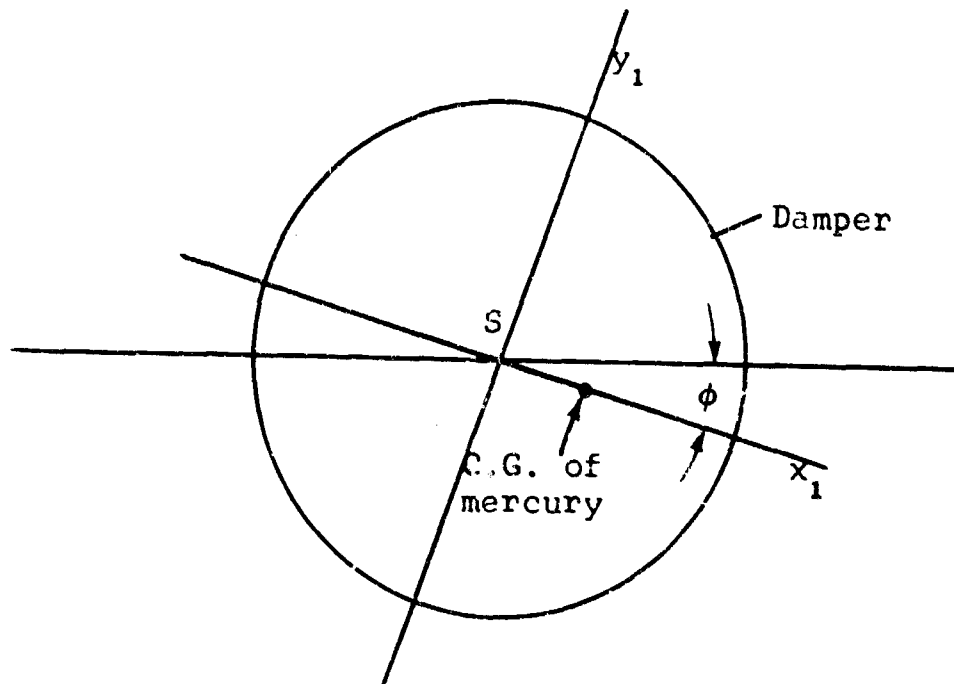


The torque vector due to the centrifugal force field, calculated for the origin 0 is then, noting Eq. 6,

$$\begin{aligned} \tau_L &= \int_M \beta^2 (ix+jy+kz) \times (ix+jy) dm \\ &= \beta^2 \int_M \begin{vmatrix} i & j & k \\ x & y & z \\ x & y & 0 \end{vmatrix} dm \end{aligned}$$

$$\tau_L = \beta^2 \left(-i \int_M yz dm + j \int_M xz dm \right). \quad (A3)$$

Next let us place a set of x_1, y_1, z_1 axes with the z_1 axis coincident with the spin axis, the positive direction being outward, the origin at S and the positive x_1 axis passing through the center of gravity of the mercury (see figure on previous page and the figure below). The xyz and x_1, y_1, z_1



coordinates of any point are then related by the equations

$$\begin{aligned} x &= x_1 \cos \phi \cos \alpha + y_1 \sin \phi \cos \alpha + z_1 \sin \alpha + L \sin \alpha \\ y &= -x_1 \sin \phi + y_1 \cos \phi \\ z &= -x_1 \cos \phi \sin \alpha - y_1 \sin \phi \sin \alpha + z_1 \cos \alpha. \end{aligned} \quad (A4)$$

Substituting Eq. A4 in Eq. A3 we obtain

$$\begin{aligned} \tau_L &= -i\beta^2 \int_M [x_1^2 \sin \phi \cos \phi \sin \alpha \\ &\quad - y_1^2 \sin \phi \cos \phi \sin \alpha - x_1 y_1 (\cos^2 \phi - \sin^2 \phi) \sin \alpha] \end{aligned}$$

$$\begin{aligned}
& - x_1 z_1 \sin \phi \cos \alpha + y_1 z_1 \cos \phi \cos \alpha] dm \\
& + j \beta^2 \int_M [-x_1^2 \cos^2 \phi \sin \alpha \cos \alpha \\
& - y_1^2 \sin^2 \phi \sin \alpha \cos \alpha + z_1^2 \sin \alpha \cos \alpha \\
& - 2 x_1 y_1 \sin \phi \cos \phi \sin \alpha \cos \alpha \\
& + x_1 z_1 \cos \phi (\cos^2 \alpha - \sin^2 \alpha) + y_1 z_1 \sin \phi (\cos^2 \alpha \\
& - \sin^2 \alpha) - x_1 L \cos \phi \sin^2 \alpha - y_1 L \sin \phi \sin^2 \alpha \\
& + z_1 L \sin \alpha \cos \alpha] dm.
\end{aligned}$$

Since the $x_1 y_1$ plane and the $x_1 z_1$ plane are planes of symmetry for the mercury the various products of inertia vanish, leaving

$$\begin{aligned}
\tau_L = & - i \beta^2 \sin \phi \cos \phi \sin \alpha \int_M (x_1^2 - y_1^2) dm \\
& + j \beta^2 \left\{ \frac{1}{4} \sin 2\alpha \int_M [-(x_1^2 + y_1^2) + (y_1^2 - x_1^2) \cos 2\phi \right. \\
& \left. + 2z_1^2] dm - ML \bar{x}_1 \cos \phi \sin^2 \alpha \right\} \quad (A5)
\end{aligned}$$

where \bar{x}_1 is the x_1 coordinate of the center of gravity of the mercury. \bar{y}_1 and \bar{z}_1 are zero because of symmetry. We shall next digress long enough to compute the relevant integrals using the procedure and cylindrical coordinate system already used in Eqs. 87 and 90. In so doing we shall omit terms of order higher than one in b .

We have

$$\begin{aligned}
\int_M (x_1^2 - y_1^2) dm = & \rho \int_{-z_1}^{z_1} \int_0^{2\pi} \int_{a-b}^{r_2} r^2 (\cos^2 \theta \\
& - \sin^2 \theta) r dz d\theta dr
\end{aligned}$$

$$\begin{aligned} & \cong \frac{\rho}{4} \int_{-z_1}^{z_1} \int_0^{2\pi} \left\{ r_2^4 - a^4 + 4a^3b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} \cos 2\theta dz d\theta \\ & = 0, \end{aligned} \quad (A6)$$

$$\begin{aligned} \int_M (x_1^4 + y_1^4) dm &= \rho \int_{-z_1}^{z_1} \int_0^{2\pi} \int_{a-b}^{r_2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta r^3 dz d\theta dr \\ &\cong \frac{\rho}{4} \int_{-z_1}^{z_1} \int_0^{2\pi} \left\{ r_2^4 - a^4 + 4a^3b \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} dz d\theta \\ &= \frac{\rho\pi}{2} \int_{-z_1}^{z_1} (r_2^4 - a^4) dz \\ &= \rho\pi z_1 (r_2^4 - a^4), \end{aligned} \quad (A7)$$

$$\begin{aligned} \int_M z_1^2 dm &= \rho \int_{-z_1}^{z_1} \int_0^{2\pi} \int_{a-b}^{r_2} \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta z^2 r dz d\theta dr \\ &\cong \frac{\rho}{2} \int_{-z_1}^{z_1} \int_0^{2\pi} z^2 \left\{ r_2^2 - a^2 + 2ab \left[1 - \left(\frac{z}{z_1} \right)^2 \right] \cos \theta \right\} dz d\theta \\ &= \rho\pi (r_2^2 - a^2) \int_{-z_1}^{z_1} z^2 dz \\ &= \frac{2}{3} \rho\pi (r_2^2 - a^2) z_1^3. \end{aligned} \quad (A8)$$

Equations A6, A7, and A8 in A5 now give

$$\begin{aligned} \tau_L &\cong j \beta^2 \left[\frac{1}{12} \sin 2\alpha \rho\pi (r_2^2 - a^2) z_1 (4z_1^2 - 3r_2^2 - 3a^2) \right. \\ &\quad \left. - ML\bar{x}_1 \cos \phi \sin^2 \alpha \right]. \end{aligned} \quad (A9)$$

From Eq. 291 we see that b and ξ are of the same order; hence since

$$\xi = L \sin \alpha \quad (A10)$$

it follows that ξ , b , and $\sin \alpha$ are of the same order, and the last term in the bracket of Eq. A9 can be omitted. Substituting from Eq. 89 we then obtain finally

$$\tau_L \cong j \frac{\beta^2 M}{24} \sin 2\alpha (4z_1^2 - 3r_2^2 - 3a^2). \quad (A11)$$

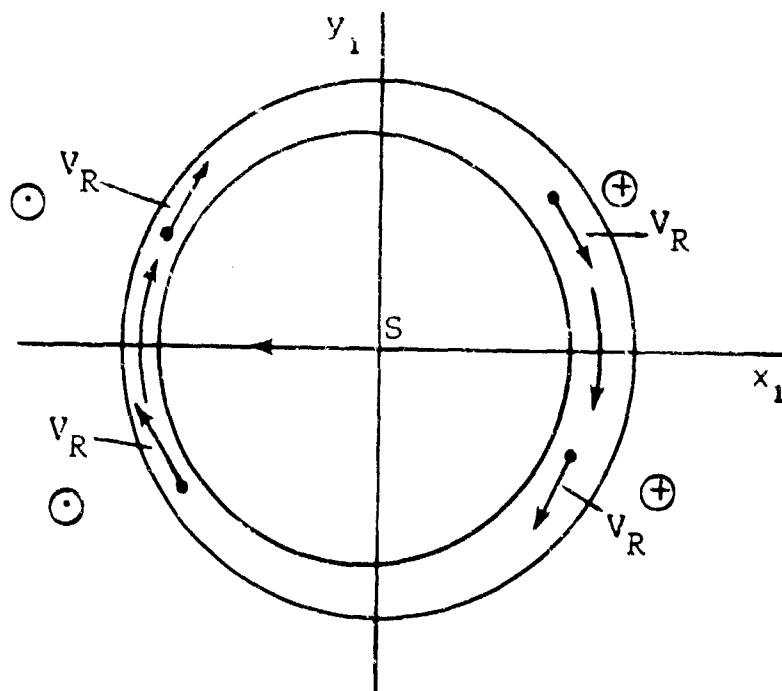
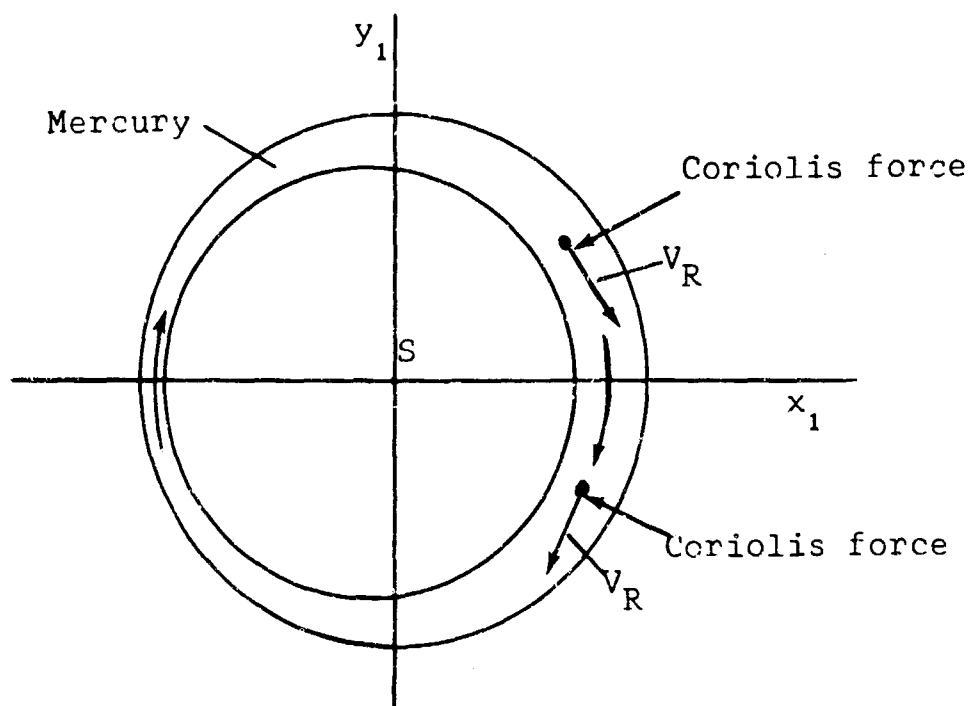
Determination of τ_C . First we note that since $F_C = 0$, as indicated by Eq. A2, the torque vector due to the Coriolis force field is independent of the point for which it is calculated. We may therefore choose S as the point for which we calculate τ_C . Next we note that since the cross product is distributive τ_C may be regarded as the sum of the contributions of the following components into which the precessional angular velocity β is resolved.

1. $\beta \cos \alpha$ in the $+z_1$ direction,
2. $\beta \sin \alpha \cos \phi$ in the $-x_1$ direction, (A12)
3. $\beta \sin \alpha \sin \phi$ in the $-y_1$ direction.

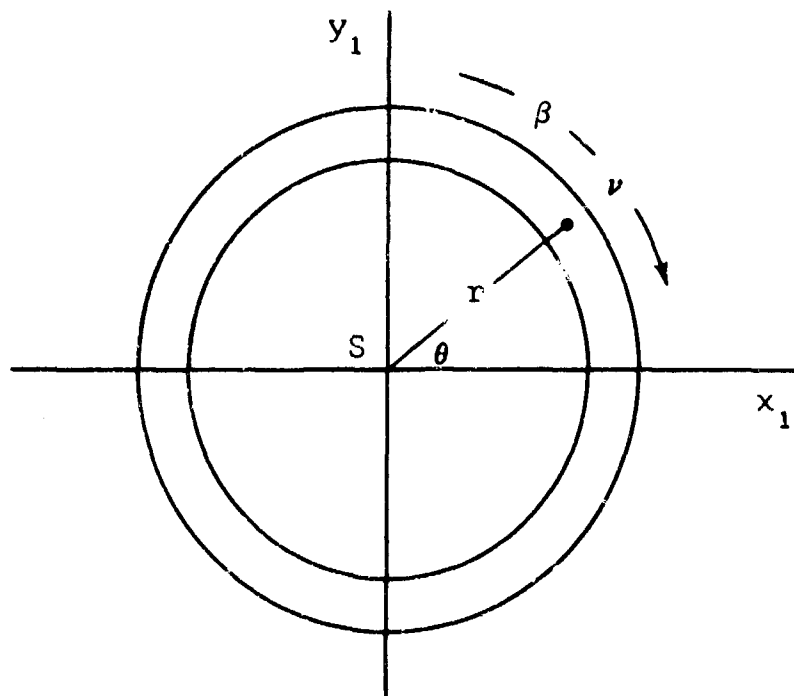
We shall now compute these contributions.

The Coriolis forces on two symmetrically placed equal elements of mass dm due to the first of the components (Eq. A12) are shown in the figure at the top of the following page. It is now evident that the four Coriolis forces which act on four equal elements of mass dm symmetrically located at points $(x_1, \pm y_1, \pm z_1)$ together have a resultant which passes through S , and hence contributes nothing to the torque vector for this point. Since the mercury may be regarded as composed of such sets of symmetrically placed mass elements it follows that the contribution of the first (and largest) of the three angular velocity components (Eq. A12) to τ_C is zero.

Turning to the second of the components in Eq. A12 we have the situation shown in the figure at the bottom of the following page, where \odot indicates "up" and \oplus indicates "down". The resultant of these forces is a couple whose vector points



in the $+y_1$ direction and is the desired contribution to τ_C . We wish to determine this accurate through first-order terms in b . But the angular velocity component has a factor $\sin\alpha$, which, we saw above, is of first order in b ; hence in computing the moment of the Coriolis force system about the y_1 axis we may place $b=0$ in the expressions which give V_R and the shape of the mercury. In this way we obtain directly the desired first term in the Maclaurin series expansion of the moment. Under these circumstances we see the mercury as an annular ring with no eccentricity rotating clockwise with angular velocity $(\beta-\nu)$, as shown below.



Using cylindrical coordinates for the integration we obtain for the desired moment

$$\begin{aligned} \text{Moment} &\cong 2\rho\beta\sin\alpha\cos\phi \int_{-z_1}^{z_1} \int_0^{2\pi} \int_a^{r_2} (\beta-\nu)r\cos\theta\cos\theta r dz d\theta dr \\ &= \frac{1}{2} \rho\beta(\beta-\nu) \sin\alpha\cos\phi (r_2^4 - a^4) \pi 2z_1 \end{aligned}$$

$$\text{Moment} \cong \frac{1}{2} M\beta(\beta-\nu)(r_2^2 + a^2)\cos\phi \sin\alpha. \quad (A13)$$

The contribution of the second component in Eq. A12 to τ_C is a vector having the $+y_1$ direction and the magnitude (Eq. A13).

The third component in Eq. A12 can be treated in the same manner as the second. Since the two are perpendicular, and one has a factor $\sin\phi$ whereas the other has a $\cos\phi$, corresponding modifications must be made in the contribution to τ_C . Noting Eq. A13 it follows that the contribution of the third component of Eq. A12 to τ_C is a vector in the $-x_1$ direction of magnitude

$$\text{Magnitude} \cong \frac{1}{2} M\beta(\beta-\nu)(r_2^2+a^2)\sin\phi \sin\alpha. \quad (\text{A14})$$

If these two vector contributions are resolved into components along the x, y, and z directions and combined, we find that τ_C consists of a vector extending in the $+y$ direction and given by

$$\tau_C \cong j \frac{1}{2} M\beta(\beta-\nu)(r_2^2+a^2)\sin\alpha. \quad (\text{A15})$$

In view of Eqs. A1 and A2 we see that the expressions in Eq. 311 for $|F_1|$ and $|F_2|$ remain valid in the present case. We note that these are of first order in ξ and hence b . It is for this reason that expressions accurate through first order-terms in b are adequate in computing torque vectors. The torque vectors τ_L and τ_C , given by Eqs. A11 and A15, respectively, are evidently provided for by letting the force component vector F_1 act at W as before; but moving the point of application of the component vector F_2 to a point on the precession axis whose distance from W is

$$\begin{aligned} \text{Distance from } F_2 \text{ to W} &\cong - \frac{j \text{ component of } (\tau_L + \tau_C)}{|F_2|} \\ &\cong \frac{1}{2L} \left(1 + \frac{2a^2\beta^2\sin\phi\cos\phi}{3(r_2^2-a^2)K\mu(\beta-\nu)} \right)^{-1} \\ &\quad \times \left\{ \frac{1}{6} (3r_2^2+3a^2-4z_1^2) - \left(1 - \frac{\nu}{\beta} \right) (r_2^2+a^2) \right\} \end{aligned} \quad (\text{A16})$$

toward the center of gravity of the main spinning body. Here $|F_2|$ is given by Eq. 311.

We have seen that the torque on the main spinning body due to F_1 is of importance in determining the damping factor α and the rate of increase of the spin velocity $\dot{\nu}$, whereas the torque due to F_2 is of importance in determining the shift in the precessional angular velocity β . We have not altered F_1 , and have changed only the point of application of F_2 . The following final results are now evident. The three dimensional nature of the damper and its tilt require no correction whatever in the damping factor α or the rate of increase in spin velocity $\dot{\nu}$. The shift in precessional angular velocity is affected by this refinement, and can be gotten by replacing L by [L-distance given by Eq. A16] in the expression for k_y in Eq. 313 before using it in Eq. 317. We are really interested, however, only in the damping factor.

Appendix B

OUTLINE OF THE PROCEDURE FOR SOLVING A PROBLEM

A. Eccentric Annular Configuration

1. Determine a , the inner radius of the mercury with no wobble. This can be done using Eq. 89.
2. Determine K using Eq. 280.
3. Tentatively placing

$$\beta = \frac{A\nu}{B} \quad (B1)$$

determine \bar{P} from Eq. 298, and \bar{Q} from Eq. 299.

4. Determine ϕ from Eq. 303 and the table on page 94.
5. Determine k from Eq. 314; then determine k_x , k_y , and k_z from Eq. 313.
6. Obtain $\dot{\nu}$ from Eq. 315, $\left(\frac{A\nu}{B} - \beta\right)$ from Eq. 317, and α from Eq. 318.

If desired the calculated value of $\left(\frac{A\nu}{B} - \beta\right)$ can be used to give an improved value of β , which can then be used to replace Eq. B1; and the subsequent calculations repeated, and so on.

B. Crescent Configuration

1. Determine θ_m from Eq. 344.
2. Determine r_m , the mean radius, from the relation

$$r_m = \frac{1}{2} (r_1 + r_2) . \quad (B2)$$

3. Determine K_0 from Chézy's formula, which is described in the reference (footnote B) on page 106, and in other books on applied hydrodynamics.
4. Tentatively placing

$$\beta = \frac{A\nu}{B} \quad (B3)$$

$\dot{\nu}$ is given by Eq. 355, the motion of the spin axis is given by Eq. 357, and $\left(\frac{A\nu}{B} - \beta\right)$ is given by Eq. 358, in which ξ and $\ddot{\xi}$ are given by Eqs. 357 and 359, respectively.

If desired the calculated value of $\left(\frac{A\nu}{B} - \beta\right)$, which is now a function of t , can be used to suggest a value of β which supersedes that given by Eq. B3, after which the subsequent calculations are repeated; and so on.

The time at which the wobble would disappear if the crescent configuration did not go over into the eccentric annular configuration is given by placing $\xi=0$ in Eq. 357 and solving for t . Actually, as stated on page 109, the crescent configuration cannot persist for values of ξ which are smaller than that given by equating $-y_0$, given by Eq. 350 and \bar{x} , given by Eq. 346. The value of t at which this value of ξ is reached is given by Eq. 357.

NOTE. In this report the analysis of the eccentric annular case is intended for use only in connection with large mercury dampers, wherein surface tension plays but a small role. (See footnote 5, page 79.) In any particular case it is hence necessary that the terms in the calculations which are due to surface tension be relatively small.